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A Class of Distortion Operators for Pricing Financial and Insurance Risks

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Abstract

This article introduces a class of distortion operators, $g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha]$, where Φ is the standard normal cumulative distribution. For any loss (or asset) variable X with a probability distribution $S_X(x) = 1-F_X(x)$, $g_\alpha[S_X(x)]$ defines a distorted probability distribution whose mean value yields a risk-adjusted premium (or an asset price). The distortion operator g_α can be applied to both assets and liabilities, with opposite signs in the parameter α. Based on CAPM, the author establishes that the parameter α should correspond to the systematic risk of X. For a normal (μ,σ^2) distribution, the distorted distribution is also normal with $\mu' = \mu + \alpha\sigma$ and $\sigma' = \sigma$. For a lognormal distribution, the distorted distribution is also lognormal. By applying the distortion operator to stock price distributions, the author recovers the risk-neutral valuation for options and in particular the Black-Scholes formula.

Introduction

This study discusses the price of risk for both insurance and financial risks. The price of an insurance risk is also called risk-adjusted premium, excluding expenses. Numerous and diverse theories exist on the price of risk in the literatures of economics, finance, and actuarial science. The objective of this study is to take a unified approach and integrate economic, financial, and actuarial pricing theories.

There are two competing economic theories for the price of risk. The expected utility theory has dominated the financial and insurance economics for the past half century. Its influence in actuarial risk theory is evident (see Borch, 1961; Bühlmann, 1980; and Goovaerts et. al., 1984). Over the past decade, a dual theory of risk has been developed in the economic literature by Yaari (1987) and others. Based on Venter's (1991) observation on insurance layer prices, Wang (1995, 1996) proposed calculating insurance premium by transforming the decumulative distribution function, which turned out to coincide with Yaari's economic theory of risk.

The first major financial pricing theory is the capital asset pricing model (CAPM).

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Built on Harry Markowitz's portfolio theory, CAPM was developed by William Sharpe, John Lintner, Jan Mossin, and others. CAPM is a set of predictions concerning equilibrium expected returns on assets. It has greatly affected our perception of risk and our ways of thinking when making investment decisions. However, CAPM has serious drawbacks when applied to insurance pricing. The CAPM assumption that asset returns are normally distributed is no longer valid for insurance if loss distributions are skewed. Another difficulty with insurance CAPM is the estimation errors associated with the underwriting beta (see Cummins and Harrington, 1985).

Another centerpiece of the financial pricing paradigm is option-pricing theory. Over the past two decades, the financial field has witnessed tremendous growth of activities using options and other derivatives. The wide acceptance of the Black-Scholes formula contributed to this financial revolution. Some researchers noted the resemblance between an option and a stop-loss reinsurance cover, which called for an analogous approach to pricing insurance risks. Unfortunately, the Black-Scholes formula applies only to lognormal distributions, while actuaries work with a large array of distribution forms. Furthermore, there are significant differences between option pricing and actuarial pricing. Mildenhall (1999) provides an excellent discussion of the differences between these two approaches. Option-pricing methodology defines prices as the minimal cost of setting up a hedging portfolio, while actuarial pricing is based on the actuarial present value of costs and the law of large numbers. Using financial jargon, option pricing is done in a world of Q-measure, whereas actuarial pricing is done in a world of P-measure.

In an age in which financial and insurance risks are becoming more integrated, it is highly desirable to have a unified pricing theory. Many researchers, including Smith (1986), Cummins (1990, 1991), Embrechts (1996), and others, have expressed this viewpoint. Considerable efforts have been made by actuaries and financial economists to connect financial and insurance pricing theories (see D'Arcy and Doherty, 1988, and Gerber and Shiu, 1994). Although researchers are still trying to put together various pieces of pricing theory puzzles, an overall picture has not yet emerged.

In the actuarial literature on the price of risk, the proportional hazards (PH) transform is gaining the attention of actuaries. The PH-transform, as a special member of the general class of Wang (1996), exhibits many desirable properties, especially in pricing insurance layers. However, the PH-transform fails to reproduce the Black-Scholes formula for lognormal risks. Moreover, the PH-transform cannot be applied simultaneously to assets and liabilities.

This article proposes a new distortion operator in the general class of Wang (1996). Unlike the PH-transform, this new distortion operator is equally applicable to assets and losses. For stop-loss reinsurance covers, this distortion operator resembles a risk-neutral valuation of financial options. This distortion operator connects four different approaches: (i) the traditional actuarial standard deviation loading principle, (ii) Yaari's economic theory of risk, (iii) CAPM, and (iv) option-pricing theory.

The flow of this article is as follows: The "Distortion Operator and Insurance Pricing" section introduces the concept of a distortion operator within the context of insurance layer pricing. The "Choquet Pricing of Assets and Losses" section discusses the pricing of assets and losses using distortion operators. The next section introduces a new distortion operator and discusses its properties. In "The Implied α From Asset Prices" section, the author derives the implied distortion parameter from asset prices. In "The Parameter α and Systematic Risk" section, based on the capital asset pricing model, the author shows that the distortion parameter should correspond to the systematic risk. In the "Recovery of the Black-Scholes Formula" section, by applying the new distortion operator to stock price distributions, the author recovers a risk-neutral valuation of options, in particular the Black-Scholes formula. The next section discusses the fundamental difference between a distortion operator and a transformed distribution. The "Measure of Downside Risk and Tail Thickness" section discusses some related measures of downside risk and tail thickness. The following section discusses some practical issues in pricing insurance, and the final section gives two examples of pricing insurance using the new distortion operator.

DISTORTION OPERATOR AND INSURANCE PRICING

Let X be a non-negative loss random variable with cumulative distribution function $F_X(x) = P(X \le x)$. The decumulative distribution function, denoted by $S_X(x) = 1 - F_X(x)$, has a special role in calculating insurance premiums based on the fact that

$$E[X] = \int_0^\infty S_X(y)dy.$$

An insurance layer $X(a, a+m]$ is defined by a payoff function

$$X(a,a+m] = \begin{cases} 0, & \text{when } 0 \le X < a, \\ X-a, & \text{when } a \le X < a+m, \\ m, & \text{when } a+m \le X, \end{cases}$$

where a is the attachment point (also called deductible or retention) and m is the limit. The decumulative distribution function for the layer $X(a, a+m]$ is related to that of the underlying risk X by the following equation:

$$S_{X(a,a+m]}(y) = \begin{cases} S_X(a+y), & \text{when } 0 \le y < m, \\ 0, & \text{when } m \le y. \end{cases}$$

The expected loss for the layer $X(a,a+m]$ can be calculated by

$$E[X(a,a+m]] = \int_0^\infty S_{X(a,a+m]}(y)dy = \int_a^{a+m} S_X(x)dx.$$

For a very small layer $X(a, a+\varepsilon]$, the net premium (expected loss) is $S_X(a)\cdot\varepsilon$. This explains why S_X is also called the "layer net premium density." Lee (1988) gives a detailed account of S_X in relation to expected layer loss cost.

Venter (1991) showed that, for any given risk, market prices by layer always imply a transformed distribution. Inspired by Venter's insightful observation, Wang (1996) suggested calculating premium by directly transforming the decumulative distribution function:

$$H_g[X] = \int_0^\infty g[S_X(x)]dx. \tag{1}$$

The function g: [0, 1] → [0, 1] is an increasing function with $g(0) = 0$ and $g(1) = 1$. This study refers to g as a *distortion operator*. A distortion operator transforms a probability distribution S_X to a new distribution $g[S_X]$. The mean value under the distorted distribution, $H_g[X]$, represents risk-adjusted premium, excluding acquisition or internal expenses.

It is obvious that

$$H_g[X(a, a+m]] = \int_0^\infty g[S_{X(a,a+m]}(y)]dy = \int_a^{a+m} g[S_X(x)]dx.$$

For layer X(a,a+m] the risk-adjusted premium is the same whether (i) the layer X(a, a + m] is treated as a stand-alone risk and g is applied to its decumulative distribution function $S_{X(a,a+m]}(y)$ or (ii) the ground-up loss distribution is transformed to $g[S_X(x)]$, from which we calculate the expected loss to the layer.

As demonstrated in Wang (1996), the desirable distortion operator for pricing insurance layers should meet the following criteria:

- $0 < g(u) < 1$, $g(0) = 0$, and $g(1) = 1$. These conditions ensure that (i) for each value of x, $g[S_X(x)]$ defines a valid probability and (ii) *non*-zero probability events will still have (*non*-)zero probability after applying the distortion operator g.

- $g(u)$ is an increasing function (where it exists, $g'(u) \geq 0$). This is to ensure that (i) the distorted probability $g[S_X(x)]$ defines another distribution and (ii) the risk-adjusted layer premium decreases as the layer increases for fixed limit.

- $g(u)$ is concave (where it exists, $g''(u) \geq 0$). This is to ensure that (i) the risk load is non-negative for every risk or layer and (ii) the relative risk loading increases as the attachment point (retention) increases for a fixed limit.

- $g'(0) = +\infty$. This is needed to ensure unbounded relative loading at extremely high layers. Unbounded relative loading at high reinsurance layers seems to be supported by observed market reinsurance premiums (see Venter, 1991). Butsic (1999) also showed that the loss beta is unlimited at very high layers.

Wang (1996) considered a number of elementary one-parameter functions and concluded that only the power function $g(u) = u^r$, $(0 < r \leq 1)$ satisfied all these requirements. The power function corresponds to the PH-transform in Wang (1995). Wang, Young, and Panjer (1997) give a characterization of the PH-transform by an axiom regarding evaluation of compound Bernoulli risks. Although the PH-transform has some unique and desirable characteristics, researchers and practitioners have expressed some concerns, enumerated as follows:

1. The PH-transform of a lognormal distribution is no longer a lognormal distribution. To some this is a bit of a disappointment since it does not yield an analogy to the Black-Scholes formula for pricing financial options.

2. The PH-transform has a very simple functional form. However, this simplicity also comes with a limitation in terms of flexibility in its shape. To some insurance market price observers, the PH-transform sometimes yields a relative loading that increases too fast at high layers.
3. The PH-transform cannot be applied simultaneously to both assets and liabilities, as explained in the next section.

CHOQUET PRICING OF ASSETS AND LOSSES

With a broader perspective, we allow a loss variable X to be negative to include assets, and allow an asset variable A to be negative to include losses. For a consistent valuation, an asset A can be viewed as a negative loss $X = -A$ and vice versa.

For any variable X with decumulative distribution function $S_X(x)$, $(-\infty < x < \infty)$, the Choquet integral with respect to distortion operator g is defined by

$$H_g[X] = \int_{-\infty}^{0} \{g[S_X(x)] - 1\}\, dx + \int_{0}^{\infty} g[S_X(x)]\, dx. \tag{2}$$

Several authors, including Yaari (1987); Wang (1996); Wang, Young, and Panjer (1997); and Chateauneuf et al., (1996), suggested using the Choquet integral as a general pricing framework.

Definition 1. For a risk X and a real-valued function h, we say that $Y = h(X)$ is a derivative of X, since the payoff of Y is a function of the outcome of X. If the function h is nondecreasing, we say that Y is a comonotonic derivative of the underlying risk X.

Theorem 1. When using the Choquet integral H_g to price a comonotonic derivative $Y = h(X)$ of risk X, the following two methods are equivalent:

- Distortion Method: treat Y as a stand-alone risk and apply g to S_Y directly:

$$H_g[Y] = \int_{-\infty}^{0} \{g[S_Y(y)] - 1\}dy + \int_{0}^{\infty} g[S_Y(y)]dy.$$

- Transformation Method: first apply the distortion operator g to the distribution of the underlying risk X that $S_{X'}(x) = g[S_X(x)]$; then evaluate the expected value of $Y = h(X')$ under the transformed (ground-up) distribution $S_{X'}$.

Theorem 1 shows that the Choquet integral H_g facilitates a risk-neutral valuation of comonotonic derivatives. A layer $X(a, a + m]$ is a comonotonic derivative of the underlying loss variable X. Thus the Choquet integral H_g facilitates a risk-neutral valuation for insurance layers. However, the above two methods are not equivalent for derivatives that are not comonotonic with the underlying risk. For example, consider an insurance contract that pays the full loss if the loss is below a limit m, and zero payment otherwise. The payoff $Y = X$ when $X < m$, and $Y = 0$ when $X \geq m$. This contract is not a comonotonic derivative of the underlying risk. The two methods in Theorem 1 are not equivalent for pricing such a contract. The distortion method (with concave g) always produces a non-negative loading, while transforming the ground-up loss distribution can yield a negative loading. Such an example can be found in Wang and Young (1998).

For the loss variable X, this article uses concave distortion operator g so that $H_g[X] \geq E[X]$. That is, the risk-adjusted premium is no less than the expected loss value.

Through expressing an asset as a negative loss, it can be shown that (see Denneberg, 1994)

$$H_g[-A] = -H_{g^*}[A],$$

where $g^*(u) = 1 - g(1-u)$ is the *dual* distortion operator of g.

If g is concave, then g^* is convex, and

$$H_g[A] \leq E[A].$$

That is, the price of an asset is no greater than the expected asset value.

In most cases, a distortion operator g and its dual distortion operator g^* are from different parametric families. The desirable properties of g might not hold when considering g^*. A family of parametric distortion operators g may exhibit desirable properties from a loss perspective, but the same parametric family may not be appropriate from an asset perspective.

If we apply the PH-transform $g(u) = u^r$ to insurance loss distributions, then we need to apply the dual distortion operator $g^*(u) = 1 - (1-u)^r$ to asset distributions. Likewise, if we apply the PH-transform $g(u) = u^r$ to assets, then we have to apply the dual distortion operator $g^*(u) = 1 - (1-u)^r$ to losses. Using different classes of parametric distortion operators for assets and losses does not promote a unified approach to pricing financial and insurance risks. This drawback is associated with most (but not all) distortion operators. Symmetric treatments of assets and liabilities are achievable, if one chooses a particular family of distortion operators g.

A NEW DISTORTION OPERATOR

Let $\Phi(x)$ be the standard normal cumulative distribution function with a probability density function

$$f(x) = \frac{d\Phi(x)}{dx} = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ for } -\infty < x < \infty.$$

Let $x = \Phi^{-1}(u)$ denote the inverse function of $u = \Phi(x)$. We define a distortion operator as follows:

$$g_\alpha(u) = \Phi[\Phi^{-1}(u) + \alpha], \tag{3}$$

where α is a real-valued parameter. Note that g_α in equation (3) satisfies the following properties:

- The limits are

$$g_\alpha(0) = \lim_{u \to 0+} g_\alpha(u) = 0, \text{ and } g_\alpha(1) = \lim_{u \to 1-} g_\alpha(u) = 1.$$

- The first derivative is

$$\frac{dg_\alpha(u)}{du} = \frac{f(x+\alpha)}{f(x)} = e^{-\alpha x - \alpha^2/2} > 0.$$

- The second derivative is

$$\frac{d^2 g_\alpha(u)}{du^2} = \frac{-\alpha f(x+\alpha)}{f(x)^2}.$$

Thus, g_α is concave ($g_\alpha'' > 0$) for positive α, and convex ($g_\alpha'' > 0$) for negative α.

- For $\alpha > 0$,

$$g_\alpha'(0) = \lim_{u \to 0+} \frac{dg_\alpha(u)}{du} = \lim_{x \to -\infty} e^{-\alpha x - \alpha^2/2} = +\infty.$$

- The dual distortion operator of g_α is

$$g_\alpha^*(u) = 1 - g_\alpha(1-u) = g_{-\alpha}(u).$$

  In other words, the dual distortion operator of g_α can be obtained by simply changing the sign of α. This property is due to the symmetry of the standard normal distribution around the origin.[1]

Therefore, for $\alpha > 0$, g_α meets all necessary criteria as listed for a desirable distortion operator.

Definition 2. For the distortion operator $g_\alpha = \Phi[\Phi^{-1}(u) + \alpha]$, a special notation is designated for the Choquet integral:

$$\mathrm{H}[X;\alpha] = \int_{-\infty}^{0} \{g_\alpha[S_X(x)] - 1\}dx + \int_0^\infty g_\alpha[S_X(x)]dx. \tag{4}$$

Building on the properties for the general class in Wang (1996), the properties of $\mathrm{H}[X;\alpha]$ are summarized as follows:

- $\min[X] \le \mathrm{H}[X;\alpha] \le \max[X]$.
- $\mathrm{H}[X;\alpha]$ is an increasing function of α. As α increases from $-\infty$ to $+\infty$, $\mathrm{H}[X;\alpha]$ increases from $\min[X]$ to $\max[X]$.
- For the constant c, $\mathrm{H}[c;\alpha] = c$ and $\mathrm{H}[X+c;\alpha] = \mathrm{H}[X;\alpha] + c$.
- For the nonconstant variable X, $\mathrm{H}[X;\alpha] < \mathrm{E}[X]$ if $\alpha < 0$; $\mathrm{H}[X;\alpha] = \mathrm{E}[X]$ if $\alpha = 0$; and $\mathrm{H}[X;\alpha] > \mathrm{E}[X]$ if $\alpha > 0$.
- For $b > 0$, $\mathrm{H}[bX;\alpha] = b\,\mathrm{H}[X;\alpha]$.

[1] Note that $1 - \Phi(x) = \Phi(-x)$ and $\Phi^{-1}(1-u) = -\Phi^{-1}(u)$. For $u = \Phi(x)$, $\Phi^{-1}(1-u) = -x$, and $g_{\alpha^*}(u) = 1 - \Phi[\Phi^{-1}(1-u) + \alpha] = 1 - \Phi(-x+\alpha) = g_{-\alpha}(u)$.

- For $b < 0$, $H[bX;\ \alpha] = b\,H[X; -\alpha]$. As a special case, $H[-X;\ \alpha] = -H[X; -\alpha]$.
- If X_1 and X_2 are comonotone[2], then $H[X_1+X_2;\ \alpha] = H[X_1;\ \alpha] + H[X_2;\ \alpha]$. Two layers of the same risk are comonotone, and thus

$$H[X(a,b];\alpha] + H[X(b,c];\alpha] = H[X(a,c];\alpha], \text{ where } a < b < c.$$

- For any two variables X_1 and X_2, $H[X_1+X_2;\ \alpha] \le H[X_1;\ \alpha] + H[X_2;\ \alpha]$, if $\alpha > 0$; and $H[X_1+X_2;\ \alpha] \ge H[X_1;\ \alpha] + H[X_2;\ \alpha]$, if $\alpha < 0$. These inequalities indicate the benefit of diversification, with the exception that there is no diversification between comonotonic risks.
- With positive α for losses (or negative α for assets), $H[X;\ \alpha]$ preserves the first and second order stochastic dominance (see Rothschild and Stiglitz, 1970).
- For a Bernoulli (Θ) risk with $P\{X = 1\} = \Theta$, if $\alpha > 0$,

$$\lim_{\Theta \to 0} \frac{H[X;\alpha]}{E[X]} = g'_\alpha(0) = +\infty.$$

If risk X has a Normal(μ, σ^2) distribution with decumulative distribution function S_X, then $S_{X'} = g[S_X]$ is another normal distribution with $\mu' = \mu + \alpha\sigma$ and $\sigma' = \sigma$. Thus, $H[X;\ \alpha] = E[X] + \alpha\,\sigma[X]$. This recovers the traditional standard deviation premium principle, where even the parameter α remains the same for both methods.[3]

If risk Y with distribution S_Y has a lognormal distribution such that $\ln(Y) \sim$ Normal(μ, σ^2), then $S_{Y'} = g[S_Y]$ is another lognormal distribution with $\mu' = \mu + \alpha\sigma$ and $\sigma' = \sigma$.

Stock prices are often modeled by lognormal distributions (so stock returns follow normal distributions). Results are equivalent whether distortion operator g_α is applied to the stock price distribution or to the stock return distribution.

The distortion operator g_α can be applied to any probability distribution. Although there is no closed formula for g_α, numerically g_α is fairly easy to calculate on a computer. Many computer languages have both Φ and Φ^{-1} as built-in functions. For instance, in Microsoft Excel, $\Phi(y)$ can be evaluated by NORMDIST(y, 0, 1, 1) and $\Phi^{-1}(y)$ can be evaluated by NORMINV(y, 0, 1).

In modeling correlation between risks, one of the most flexible models is the normal copula, which involves the functions Φ and Φ^{-1} (see Frees and Valdez, 1998; Wang, 1998b). By the same token, the distortion operator g_α can also be a practical tool for calculating risk load. The distortion operator g_α can be generalized to multivariate applications by applying g_α to their joint cumulative distribution function.

THE IMPLIED α FROM ASSET PRICES

It is assumed that assets can be priced by applying $H[X; -\alpha]$ to the present value of the future asset price at some moment in time. From current asset price and the future price distribution an implied α can be derived.

[2] There exist a variable Z and nondecreasing functions f_1 and f_2 such that $X_1 = f_1(Z)$ and $X_2 = f_2(Z)$. For a more detailed discussion on comonotone risks, see Dennenberg (1994).

[3] If X is normal, then for any distortion function g, $H_g[X]$ reduces to the standard deviation principle. With the specific distortion operator g_α, it happens that α is the same as the parameter used in the standard deviation principle.

One-Period Horizon

Assume a time horizon of one year. Consider an asset (stock) i with current price $A_i(0)$ and prospective ending period price $A_i(1)$. Let $R_i = A_i(1)/A_i(0) - 1$ denote the per annum return compounded annually. Assume that R_i has a normal distribution with mean $E[R_i]$ and standard deviation $\sigma[R_i]$.

Assume that the current stock price, $A_i(0)$, can be derived by applying $H[X; -\alpha_i]$ to the present value of ending period stock price, $A_i(1)$. We have

$$A_i(0) = H\left[A_i(1)/(1+r_f); -\alpha_i\right] = H\left[A_i(0)(1+R_i)/(1+r_f); -\alpha_i\right],$$

where r_f is the per annum risk-free rate compounded annually.

As a result, the risk-adjusted rate of return for stock i must be the same as the risk-free rate:

$$H[R_i; -\alpha_i] = E[R_i] - \alpha_i \sigma[R_i] = r_f,$$

which implies that

$$\alpha_i = \frac{E[R_i] - r_f}{\sigma[R_i]}. \tag{5}$$

Note that an asset i can also refer to an asset portfolio. For the market portfolio, M, the risk-adjusted rate of return must equal the risk-free rate:

$$r_f = H[R_M; -\alpha_M] = E[R_M] - \alpha_M \sigma[R_M],$$

which implies that

$$\alpha_M = \frac{E[R_M] - r_f}{\sigma[R_M]}$$

The right-hand side of the above equation is also called the market price of risk (see Cummins, 1990, p. 135).

A Multi-Period Horizon

Now the time horizon is extended from 1-period to T-period. Without loss of generality, assume that the current time is $t = 0$. For an asset (stock) i, let R_{it}, $t = 1, 2, \ldots, T$, denote the per annum return compounded annually in the time period i. Let $A_i(t)$ be the price of stock i at time t. We have for $t = 1, 2, \ldots, T$,

$$R_{it} = \ln A_i(t) - \ln A_i(t-1).$$

In financial economics, it is commonly assumed that R_{it} and R_{is} are independent for different time periods t and s. For simplicity, assume that R_{it} has constant mean and standard deviation in all time periods; that is, $E[R_{it}] = E[R_i]$ and $\sigma[R_{it}] = \sigma[R_i]$, for $t = 1, 2, \ldots, T$.

Let R_{it} denote the total T-period return for stock i (without compounding). Therefore,

$$R_i(T) = \ln A_i(T) - \ln A_i(0) = \sum_{t=1}^{T} \{\ln A_i(t) - \ln A_i(t-1)\} = \sum_{t=1}^{T} R_{it},$$

with

$$\mathrm{E}[R_i(T)] = \sum_{t=1}^{T} E[R_{it}] = T \cdot \mathrm{E}[R_i],$$

and

$$\mathrm{Var}[R_i(T)] = \sum_{t=1}^{T} \{\sigma[R_{it}]\}^2 = T \cdot \{\sigma[R_i]\}^2.$$

The total T-period return $R_i(T)$ for stock i has mean = $T \cdot \mathrm{E}[R_i]$ and standard deviation = $T^{1/2} \cdot \sigma[R_i]$. Assuming that for some α_i the risk-adjusted rate of return for stock i, $\mathrm{H}[R_i(T); -\alpha_i]$ equals the total T-period risk-free rate, $T \cdot r_f$, producing

$$T \cdot \mathrm{E}[R_i] - \alpha_i \sqrt{T} \sigma[R_i] = T \cdot r_f,$$

and thus

$$\alpha_i = \sqrt{T} \left\{ \frac{\mathrm{E}[R_i] - r_f}{\sigma[R_i]} \right\}. \tag{6}$$

The implied parameter α_i increases as the time horizon T increases; more precisely, α_i is proportional to the square root of T.

Note that one can progressively refine the 1-period from one year to one quarter, one month, one day, and so on, while keeping the T-period fixed at one year. By progressively refining the time periods, one eventually approaches a geometric Brownian motion model for the asset price movement. In a geometric Brownian motion model, however, there is a need to re-interpret R_i as an instantaneous rate of return compounded continuously.

Continuous Time Asset Price Model

In a continuous time model, stock prices are assumed to follow a geometric Brownian motion (GBM). Consider a stock (or stock index) i. The stock price $A_i(t)$ satisfies the following stochastic differential equation:

$$\frac{dA_i(t)}{A_i(t)} = \mu_i dt + \sigma_i dW_i, \tag{7}$$

where dW_i is a random variable drawn from a normal distribution with mean equal to zero and variance equal to dt. In equation (7), μ_i is called the expected rate of return for the stock, and σ_i is called the volatility of the stock return. Let $A_i(0)$ be the current stock price at time zero. For any future time T, the prospective stock price $A_i(T)$ as defined in equation (7) has a lognormal distribution (see Hull, 1997, p. 229):

$$\ln A_i(T) - \ln A_i(0) \sim \text{Normal}\left[(\mu_i - 0.5\sigma_i^2)T, \sigma_i^2 T\right]. \tag{8}$$

Next we apply the pricing formula H[X; $-\alpha_i$] to the present value of future stock price $A_i(T)$.

For any fixed future time T, no arbitrage condition (or simply, market value concept) implies that the risk-adjusted present value of future stock price must equal the current stock price. Therefore,

$$A_i(0) = \text{H}[e^{-r_c T} A_i(T); -\alpha_{-\alpha_i}] = e^{-r_c T}\text{H}[A_i(T); -\alpha_i], \tag{9}$$

where r_c is the risk-free rate compounded continuously.

Now equation (9) is rewritten as

$$A_i(0) = e^{-r_c T}\text{E}[B_i(T)],$$

where $B_i(T)$ is drawn from a distorted distribution

$$S_{B_i(T)}(y) = g_{-\alpha_i}[S_{A_i(T)}(y)],$$

with $B_i(0) = A_i(0)$. It can be verified that

$$\ln B_i(T) - \ln B_i(0) \sim \text{Normal}\left[(\mu_i - 0.5\sigma_i^2)T - \alpha_i \sigma_i \sqrt{T}, \sigma_i^2 T\right]. \tag{10}$$

The no-arbitrage condition in equation (9) implies that

which in turn implies that

$$\alpha_i = \frac{(\mu_i - r_c)\sqrt{T}}{\sigma_i}. \tag{11}$$

The implied α_i in Equation (11) coincides with the market price of risk of asset i as defined in Hull (1997, p. 290). It is a continuous analog of the implied α_i in equation (6) under a discrete model. With the α_i in equation (11), $g_{-\alpha_i}$ transforms the asset price distribution $S_{A_i(T)}$ to a distorted distribution $S_{B_i(T)}$ with

$$\ln B_i(T) - \ln B_i(0) \sim \text{Normal}\left[(r_c - 0.5\sigma_i^2)T, \sigma_i^2 T\right], \tag{12}$$

where both the distortion parameter α_i and the expected stock return μ_i dropped out from the distorted distribution $S_{B_i(T)}$.

THE PARAMETER α AND SYSTEMATIC RISK

In the previous section, the author derived some implied α from asset prices. This section revisits the capital asset pricing model and establishes that the parameter α should correspond to the systematic risk of X with respect to the aggregate risk portfolio.

From a risk portfolio perspective, only systematic risk should be priced; this principle underlies the CAPM. In order to clarify the meaning of systematic risk, we need to specify the aggregate risk portfolio. In the stock market, systematic risk for a stock refers to its correlation with the market portfolio (a broad stock index). In the insurance market, systematic risk for a loss variable may refer to its correlation with the insurance industry aggregate loss. As pointed out by Gary Venter, given the same loss distribution, a Florida catastrophe cover is more risky than other types of insurance contracts, simply because Florida catastrophe losses are highly correlated with industry aggregate losses. The above observations suggest that, when applying the pricing formula $H[X;-\alpha_i]$, the parameter α should correspond to the systematic risk of X.

CAPM assumes that all investors have the same one-period horizon, and asset returns have multivariate normal distributions. Let R_M and $\sigma[R_M]$ be the return and standard deviation of return for the market portfolio M. The CAPM assets that

$$E[R_i] = r_f + \beta_i\left\{E[R_M] - r_f\right\},$$

where

$$\beta_i = \frac{\mathrm{Cov}[R_i, R_M]}{\{\sigma[R_M]\}^2}$$

is the beta of stock i.

The CAPM equation can be restated in a different form:

$$\frac{E[R_i] - r_f}{\sigma[R_i]} = \rho_{i,M}\left\{\frac{E[R_M] - r_f}{\sigma[R_M]}\right\}, \tag{13}$$

where

$$\rho_{i,M} = \frac{\mathrm{Cov}[R_i, R_M]}{\sigma[R_i]\cdot\sigma[R_M]}$$

is the correlation coefficient between R_i and R_M.

Equations (13) and (5) show the following relationship between the implied α_i for stock i and the implied α_M for the market portfolio:

$$\alpha_i = \rho_{i,M}\cdot\alpha_M. \tag{14}$$

In other words, the implied α_i corresponds to the systematic risk of asset i (or its correlation with the market portfolio).

Equations (13) and (14) also have the following relationship in terms of beta and the market risk premium:

$$\alpha_i\cdot\sigma[R_i] = \beta_i\cdot\{\alpha_M\cdot\sigma[R_M]\}.$$

Recall that the time horizon can be extended from 1-period to T-period. Let R_{Mt}, $t = 1, 2, \ldots, T$, be the return for the market portfolio M in time period t. Assume that for $t = 1, 2, \ldots, T$ there is a constant correlation coefficient between R_{it} and R_{Mt}:

$$\rho_{i,M} = \frac{\mathrm{Cov}[R_{it}, R_{Mt}]}{\sigma[R_{it}]\cdot\sigma[R_{Mt}]}.$$

It can be verified that

$$\mathrm{Cov}[R_i(T), R_M(T)] = \rho_{i,M} \cdot \sigma[R_i(T)] \cdot \sigma[R_M(T)],$$

and the relation $\alpha_i = \rho_{i,M}\alpha_M$ still holds for the T-period time horizon.

Recall that one can progressively refine the length of one period while keeping the T-period fixed. This refining process will eventually converge to a special case of the intertemporal, continuous time CAPM (ICAPM) of Merton (1973), where assets returns are described by geometric Brownian motions and correlations between assets are described instantaneously. More specifically, it is assumed that (i) the price movement of individual asset i can be described by the stochastic differential equation (7) and (ii) at each moment in time, the dW_i's for individual assets follow a multivariate normal distribution. Under these assumptions, the price movement of the market portfolio (a broad stock index) can be described by

$$\frac{dA_M(t)}{A_M(t)} = \mu_M dt + \sigma_M dW_M,$$

with $\rho_{i,M} = \mathrm{Cov}[dW_i, dW_M]$ being a constant over time. In this continuous time framework, the relation $\alpha_i = \rho_{i,M}\, \alpha_M$ also holds for any time horizon T.

CAPM assumes that asset returns have multivariate normal distributions. This is a reasonable assumption for asset returns. But it can be unrealistic for insurance applications in which loss distributions are highly skewed. Nevertheless, the general principle of CAPM still applies in insurance applications. Now we use $\mathrm{H}[X; \alpha]$ to extend CAPM to variables having other than normal distributions.

Note that any random variable can be transformed to a normal variable. For any variable, X, $F_X(X)$ has a uniform distribution and $\Phi^{-1}[F_X(X)]$ has a standard normal distribution.

Consider an aggregate risk portfolio

$$\{X_1, X_2, \ldots, X_k\},$$

where k is very large. The author denotes

$$Z = X_1 + X_2 + \cdots + X_k$$

as the aggregate risk. Assume that the transformed variables

$$\left\{\Phi^{-1}[F_{X_1}(X_1)], \Phi^{-1}[F_{X_2}(X_2)], \ldots, \Phi^{-1}[F_{X_k}(X_k)]\right\}$$

have a multivariate normal distribution. For any risk X,

$$\rho_{X,Z} = \mathrm{Cov}\left[\Phi^{-1}[F_X(X)], \Phi^{-1}[F_Z(Z)]\right], \tag{15}$$

is a measure of the systematic risk of X. The correlation measure $\rho_{X,Z}$ in (15) is similar to the concept of rank-order correlation (see Frees and Valdez, 1998; Wang, 1998b), and it recovers the traditional correlation coefficient for normally distributed variables.

As a generalization of CAPM, when we put forth a general pricing formula $H[X;\alpha_X]$ with $\alpha_X = \rho_{X,Z}\ \alpha_Z$ reflecting the systematic risk of X, we get an additive pricing formula for pricing individual risks. However, for skewed distributions, risk diversification by pooling individual risks may not be as effective as in the case of normal distributions. For finite insurance risk portfolios, the parameter α_X is likely to be higher than indicated by $\rho_{X,Z}\ \alpha_Z$ to reflect the residual process risk. Other important factors that need to be considered when selecting α include (i) parameter uncertainty in the estimated loss distributions, (ii) anti-selection among insurer buyers,[4] and (iii) the cost of capital commitment. With the presence of parameter uncertainty, a higher α_X may be justifiable to reflect market friction and incomplete information.

Lastly we consider the pricing of comonotonic derivatives for an underlying risk X. For an increasing function f, $Y = f(X)$ has the same level of systematic risk as X; that is $\rho_{Y,Z} = \rho_{X,Z}$. Therefore, $\alpha_X = \alpha_Y$, and the same α should be used in pricing X and its comonotone derivative Y. For the pricing formula $H[X;\alpha_X]$ with α_X reflecting the systematic risk of X, the result in Theorem 1 still holds true for risk-neutral valuation of the comonotonic derivatives of X.

RECOVERY OF THE BLACK-SCHOLES FORMULA

A European call option on the underlying stock (or stock index) i with a strike price K and exercise date T is defined by the following payoff function

$$\text{Call}(K) = \begin{cases} 0, & \text{when } A_i(T) \le K, \\ A_i(T) - K, & \text{when } A_i(T) > K. \end{cases}$$

The expected payoff for this option can be calculated as

$$E[\text{Call}(K)] = \int_0^\infty S_{Call(K)}(x)dx = \int_K^\infty S_{A_i(T)}(y)dy.$$

Being a nondecreasing function of the underlying stock price, the option payoff, Call(K), is comonotone with the terminal stock price, $A_i(T)$; thus it has the same level of systematic risk as the underlying stock i. Therefore, the same α as in equation (11) should be used to price the option Call(K), and

$$H[\text{Call}(K);-\alpha] = \int_0^\infty g_{-\alpha}\left[S_{Call(K)}(x)\right]dx = \int_K^\infty S_{B_i(T)}(y)dy.$$

In other words, the price of a European call option is the expected payoff under the distorted (risk-neutral) stock price distribution $S_{B_i(T)}$, where the expected stock return μ_i is replaced by the risk-free rate r_c. This option price is exactly the same as the Black-Scholes formula.

There is an analogy between (i) an unlimited stop-loss cover with retention K, and (ii) a European call option with strike price K. Both are comonotone derivatives of the underlying (loss or asset) variable. By applying the pricing formula $H[X;\alpha]$ to the stop-loss variable, we get a stop-loss premium that is the expected stop-loss value under a distorted ground-up loss distribution. Likewise, the price for a European call option can be evaluated as the expected option payoff under the distorted (risk-neu-

[4] As an example, relative to the average population, the mortality rates deviate differently among life annuity buyers as opposed to life insurance buyers.

tral) distribution for the underlying stock price, where the expected stock return μ_i does not appear in the option pricing model. Using $H[X;\ \alpha]$ adds a new perspective to the well-known risk-neutral valuation methodology of options (see Cox and Ross, 1976).

Several researchers, including Smith (1977) and Doherty and Garven (1986), proposed using an option-pricing model to price insurance contracts. Cummins (1988) applied the Black-Scholes formula to insurance guaranty fund premium. However, as a major limitation of the Black-Scholes formula in insurance applications, the option pricing models are not applicable for the full range of distributions used by actuaries to describe prospective losses. D'Arcy and Doherty (1988, pp. 63-64) made the following comment on the option-pricing model:

> The distribution assumptions required to use option pricing are quite specific, either normality or lognormality. These distributions may provide reasonable approximations when the underlying variable is a diversified portfolio of financial assets or policy liabilities. In this chapter we were careful to use options models that met this criterion. But consider a reinsurance policy written on a single direct policy. The reinsurance payoffs may well have the characteristics of an option. However, it would be foolhardy to use this feature as sufficient justification for pricing the reinsurance contract with an option-pricing model. If the payout on the direct policy cannot be reasonably approximated by a lognormal or normal distribution, this approach could be seriously in error.

While the Black-Scholes formula relies on the lognormal distribution assumption,[5] the pricing formula $H[X;\ \alpha]$ can be applied to any loss distributions.

As a final note, this section revisits equation (11), where the implied α increases as the time horizon lengthens. The pricing formula $H[A_i(T); -\alpha]$ with α in (11) reveals the intimate connection between Merton's intertemporal, continuous time CAPM and the option pricing theory. This interesting result may have applications in pricing long-tailed insurance when losses are not reported or settled until many years after the policy period expires. If the development of emerged losses can be modeled by geometric Brownian motions, the parameter α would be proportional to the square root of the time period from policy inception to the date of loss settlement. From another perspective, for long-tailed insurance risks, parameter α reflects both the magnitude and duration of capital commitment. The relationship (11) between α and duration T may be useful in calculating market values for insurance liabilities (including loss reserve discounting).

RELATION WITH TRANSFORMED DISTRIBUTIONS

A distortion operator g is fundamentally different from a transformed distribution $Y = h(X)$. Although every increasing transform $Y = h(X)$ can be written in the form $S_Y(x) = g[S_X(x)]$ with

$$g(u) = S_X\left(h^{-1}(S_X^{-1}(u))\right),$$

[5] Strictly speaking, this is not an accurate statement. The Black-Scholes approach can be applied to stochastic processes other than geometric Brownian motion; for example, see Gerber and Shiu (1994).

the implied g depends on the underlying distribution S_X. When applied to a different distribution $S_{X'}$, we get a different implied g.

Bühlmann (1980) proposed a premium loading method by the Esscher transform:

$$f^*(x) = \frac{e^{\alpha x} f(x)}{\mathrm{E}[e^{\alpha X}]}.$$

The Esscher transform of a Normal(μ, σ^2) distribution yields Normal($\mu + \alpha\sigma^2, \sigma^2$). Gerber and Shiu (1994) applied the Esscher transform to the logarithm of X, which can reproduce the Black-Scholes option pricing formula. Venter (1998) suggested using the log-Esscher transform as an alternative to the PH-transform for lognormal risks. However, neither the Esscher nor the log-Esscher transform corresponds to a fixed distortion operator.

Like Venter (1998), Butsic (1999) came close to conceptualizing the distortion operator g_α in equation (3). Butsic suggested a generalized PH-transform by defining

$$S_Y(x) = \left[S_X(x)\right]^{q(x)}, \text{with } 0 \le q(x) \le 1. \tag{16}$$

He considered the implied $q(x)$ by shifting the lognormal location parameter. He also considered a fractional PH-transform

$$S_Y(x) = \left[S_X(x)\right]^{\frac{qx}{x+m}} \text{ with } 0 \le m. \tag{17}$$

Note that neither formula (16) nor (17) corresponds to a distortion operator, since the implied distortion

$$g(u) = u^{q\left[S_X^{-1}(u)\right]}$$

relies on the underlying risk distribution S_X. If we replace the underlying risk distribution S_X by $S_{X'}$, then formula (16) may imply a different distortion. The same comments apply to formula (17).

As a further clarification, a transformation of variable $Z' = h(Z)$ can induce a distortion:

$$g(u) = S_Z\left(h^{-1}\left[S_Z^{-1}(u)\right]\right),$$

provided that Z and $h(\cdot)$ are kept fixed and not varying with the underlying risk distribution S_X to which g is applied. For instance, consider a simple scale transform $Z' = \rho Z$ with $0 < \rho < \infty$. It can be verified that

- if Z has an exponential distribution, then the induced distortion $g(u) = u^{1/\rho}$ gives the PH-transform;
- if Z has a lognormal distribution with $\ln(Z) \sim$ Normal(0, 1), then the induced distortion $g(u) = \Phi[\Phi^{-1}(u) + \ln(\rho)]$ is the same as g_α in equation (3) with $\alpha = \ln(\rho)$.
- if Z has a gamma distribution, then the induced distortion operator preserves gamma distributions;

- if Z has a Pareto distribution with $S_Z(z) = (1 + z)^{-\rho}$, then the induced distortion is $g(u) = \left[1 + (u^{-1/\rho} - 1) / \rho\right]^{-\rho}$. Venter (1998) discussed such a distortion function.

MEASURE OF DOWNSIDE RISK AND TAIL THICKNESS

In managing financial and insurance risks, we often need a measure of downside risk. A good measure of downside risk is essential in solvency measurement, risk-based-capital requirement, value-at-risk calculation, and dynamic financial analysis. The distortion operator g_α can also be used as a general risk measure.

Butsic (1994) advocated the use of expected policyholder deficit (EPD) as a measure of downside risk potential. He used a constant EPD ratio (to the expected loss) in deciding risk-based capital requirements. In his recent paper, Butsic (1999) suggested that the EPD should be calculated with respect to a risk-adjusted distribution. The distortion operator g_α can be used to transform any underlying distribution to a risk-adjusted distribution, from which a risk-adjusted EPD can be calculated.

Artzner et al. (1998) (also see Artzner, 1999) proposed a set of rules for a coherent risk measure, which in general would lead to a distortion operator. As a variation of the EDP concept, Artzner and his co-authors advocated a risk measure based on the expected deficit in excess of a prescribed (say, *100p*th) percentile, which corresponds to a distortion operator:

$$g(u) = \begin{cases} u, & 0 \le u < p, \\ p, & p \le u < 1. \end{cases}$$

We can modify their risk measure by using the following composite distortion operator:

$$g(u) = \begin{cases} g_\alpha(u), & 0 \le u < p, \\ g_\alpha(p), & p \le u < 1, \end{cases}$$

where g_α is defined in (3). For lognormal risks, with an appropriate value of α, this modified risk measure corresponds to the price of an option with a strike price equal to the 100 p^{th} percentile value.

Other measures of deviation can be defined utilizing the entire distribution, reflecting both upside and downside potentials. Wang (1998a) used the PH-transform (r = 0.5) to define a right-tail deviation. As an alternative, H[X; α] can be used to define measures of variability. We define a right-tail deviation and a left-tail deviation, respectively, as follows

$$\mathrm{RD}_\alpha[X] = \{\mathrm{H}[X;\alpha] - \mathrm{E}[X]\} / \alpha$$
$$\mathrm{LD}_\alpha[X] = \{\mathrm{E}[X] - \mathrm{H}[X;-\alpha]\} / \alpha,$$

for some positive α (say, $\alpha = 0.1$).

For a normal distribution Normal(μ, σ^2), we have $\mathrm{RD}_\alpha[X] = \mathrm{LD}_\alpha[X] = \sigma$.

In a financial modeling, empirical asset return data sometimes indicate a two-sided distribution, which has thicker tails than a normal distribution. Without referring to higher moments, we can define an index for the tail thickness relative to a normal tail. For an asset return variable X, we define a right-tail index and a left-tail index by

$$RTI_{\alpha}[X] = \mathrm{RD}_{2\alpha}[X] \,/\, \mathrm{RD}_{\alpha}[X],$$
$$LTI_{\alpha}[X] = \mathrm{LD}_{2\alpha}[X] \,/\, \mathrm{LD}_{\alpha}[X],$$

for some positive α (say $\alpha = 0.1$).

For a normal distribution, $RTI_{\alpha}[X] = LTI_{\alpha}[X] = 1$, since the tail deviations do not vary with α. A tail index greater than one indicates that the tail is heavier than a normal tail.

Some Practical Issues in Pricing Insurance

A prerequisite for using the pricing formula $H[X; \alpha]$ is to have an estimated loss distribution for the underlying risk. The monograph by Klugman, Panjer, and Willmot (1998) serves as an excellent source for modeling loss distributions. However, there remain a number of judgment issues related to modeling loss distributions. It is often desirable to explicitly reflect parameter uncertainty regarding frequency and severity in the estimated loss distribution. By modeling parameter uncertainty, one may incorporate knowledge or judgment beyond the underlying data. For a fixed α, the pricing formula $H[X; \alpha]$ automatically picks up an extra loading for parameter uncertainty.

In addition to frequency and severity risk, another source of uncertainty is timing risk. With prolonged duration of loss reporting and loss payments, both the investment income and the cost of capital commitment increase. Equation (11) may be useful in quantifying this intricate relationship.

The pricing formula $H[X; \alpha]$ can be applied in a number of ways, depending on the circumstances. In pricing an excess layer, one can apply $H[X; \alpha]$ to the severity distribution to derive a relativity in risk loading by layer. This approach is fairly handy, given that industry-wide severity curves for many lines of insurance are readily available from the Insurance Services Office (ISO) and the National Council on Compensation Insurance (NCCI). When pricing aggregate stop-loss contracts, one can apply $H[X;\alpha]$ to the aggregate loss distributions.

Undoubtedly the selection of α is crucial in any implementation of the pricing formula $H[X; \alpha]$. While CAPM suggests that the parameter α reflects the level of systematic risk, in insurance applications one should not estimate α based solely on statistical regressions using historical data. An alternative method for estimating the systematic risk is by employing a risk factor analysis. Such an analysis first identifies a number of key factors that influence industry aggregate losses. Examples of such key factors include conceivable major natural catastrophes, possible dramatic changes in court rulings, unexpected claim cost inflation, and sudden changes in the interest rate yield curve. The systematic risk of X can then be estimated by evaluating the sensitivity of X to these key factors. When selecting α, one should also take into consideration (i) parameter uncertainty in the estimated loss distribution,

(ii) anti-selection and moral hazards by insurance buyers, (iii) the cost of capital commitment, and (iv) competitiveness of the market.

EXAMPLES

Two numerical examples are presented in this section.

Example 1. Consider a ground-up liability risk X with a Pareto severity distribution

$$S_X(x) = \left(\frac{2000}{2000+x}\right)^{1.2}, \text{ for } x > 0.$$

To compare risk loading by layer, assume that the ground-up frequency is exactly one claim, and then apply the pricing formula H[X; α] to the severity distribution. For numerical illustration, choose a loading parameter $\alpha = 0.1$. If the loss is capped by a basic limit of \$50,000, the expected loss is \$4,793 and the risk-adjusted premium is \$5,487, implying a 14.5 percent loading. As shown in Table 1, the relative loading increases at higher layers.

A comparison can be made with the PH-transform loading method. A PH index r = 0.9245 is selected to yield the same relative loading (14.5 percent) for the basic limit layer (\$0, \$50,000). Table 1 shows that the PH-transform method produces a risk loading that increases much faster than using distortion g_α.

TABLE 1
Risk Load by Layer Under Distortion g_α and PH-transform

Layer in 000's	Expected Loss	PH Premium	Relative Loading %	H[X; α] Premium	Relative Loading %
(0, 50]	4,793	5,487	14.5	5,487	14.5
(50, 100]	657	910	38.4	845	28.6
(100, 200]	582	857	47.2	769	32.2
(200, 300]	307	475	54.7	414	34.9
(300, 400]	204	325	59.6	278	36.6
(400, 500]	150	246	63.3	207	37.8
(500, 1000]	428	728	70.1	598	39.9
(1000, 2000]	373	675	81.1	533	43.0
(2000, 5000]	420	819	94.7	616	46.5
(5000, 10000]	271	567	109.5	406	49.9

Example 2. Consider a hypothetical example of a satellite launching. Suppose that the estimated loss distribution for a commercial satellite launching is a Bernoulli type, with 5 percent probability of a total loss at \$100 million. This risk is shared by a number of (re)insurers worldwide. The satellite launching loss distribution is

$$S_X(x) = \begin{cases} 0.05, & 0 < x < 100 \\ 0, & 100 \le x. \end{cases}$$

Using the distortion operator g_α in equation (3) with $\alpha = 0.1$,

$$g_\alpha[S_X(x)] = \begin{cases} g_\alpha(0.05) = 0.0612, & 0 < x < 100 \\ 0, & 100 \leq x. \end{cases}$$

This implies a risk-adjusted premium (excluding expenses) of \$6.12 million, or a 22.4 percent loading. The large size of risk may indicate a higher systematic risk (due to higher risk concentration combined with a relatively small statistical sample for estimation). Thus a higher α may be needed for pricing this risk. If $\alpha = 0.15$ is used, the result is a risk-adjusted premium of \$6.75 million, or a 35.0 percent loading (excluding expenses).

CONCLUSION

The actuarial literature has witnessed several decades of searching for a sound pricing formula. This long search has been disjointed and along different paths. The new pricing formula $H[X;\ \alpha]$ is like a piece of connecting puzzle that ties together four different approaches: (i) the traditional standard deviation principle, (ii) Yaari's economic theory of risk, (iii) the capital asset pricing model, and (iv) option pricing theory.

Unlike other distortion operators in Wang (1996), the pricing formula $H[X;\ \alpha]$ offers a symmetric treatment of assets and losses, connects the CAPM, and recovers the Black-Scholes formula for option prices. It promotes a unified approach to pricing financial and insurance risks. The pricing formula $H[X;\ \alpha]$ has potential applications not only in increased limits ratemaking and reinsurance layer pricing, but also in pricing financial risks such as bond defaults and securitization deals.

With great promise in theoretical development and practical application, more research is needed to further explore the properties of this pricing formula.

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