Managing Financial Risk in Planning under Uncertainty

Andres Barbaro and Miguel J. Bagajewicz
School of Chemical Engineering and Materials Science, University of Oklahoma, Norman, OK 73019

DOI 10.1002/aic.10094
Published online in Wiley InterScience (www.interscience.wiley.com).

A methodology is presented to include financial risk management in the framework of two-stage stochastic programming for planning under uncertainty. A known probabilistic definition of financial risk is adapted to be used in this framework and its relation to downside risk is analyzed. Using these definitions, new two-stage stochastic programming models that manage financial risk are presented. Computational issues related to these models are also discussed. © 2004 American Institute of Chemical Engineers *AIChE J,* 50: 963–989, 2004
Keywords: planning, financial risk, robust optimization

Introduction

Planning under uncertainty is a common class of problems found in process systems engineering. Some examples widely found in the literature are capacity expansion, scheduling, supply chain management, resource allocation, transportation, unit commitment, and product design problems. The first studies on planning under uncertainty could be accredited to Dantzig (1955) and Beale (1955), who proposed the two-stage stochastic models with recourse, which provide the mathematical framework for this article.

The industrial importance of planning process capacity expansions under uncertainty has been widely recognized and discussed by several researchers (Ahmed and Sahinidis, 2000b; Berman and Ganz, 1994; Eppen et al., 1989; Liu and Sahinidis, 1996; Murphy et al., 1987; Sahinidis et al., 1989). In the majority of industrial applications, capacity expansion plans require considerable amount of capital investment over a long-range time horizon. Moreover, the inherent level of uncertainty in forecast demands, availabilities, prices, technology, capital, markets, and competition make these decisions very challenging and complex. Therefore, several approaches were proposed to formulate and solve this problem. They mainly differ in the way uncertainty is handled, the robustness of the plans, and their flexibility. This article follows the two-stage stochastic programming approach with discretization of the uncertainty space by random sampling of the parameter probability distributions. In turn, the feasibility constraints for the problem are enforced for every scenario in a deterministic fashion (taking recourse actions with an associated cost) such that the resulting plan or design is feasible under every possible uncertainty realization.

A formal two-stage stochastic model for capacity planning in the process industry was presented by Liu and Sahinidis (1996) as an extension of the deterministic models developed by Sahinidis et al. (1989). In the two-stage stochastic approach, it is assumed that the capacity expansion plan is decided before the actual realization of uncertain parameters (*scenarios*), allowing only some operational recourse actions to take place to improve the objective and correct any infeasibility. In this formulation, the objective is usually to maximize the expected profit or to minimize the expected cost over the two stages of the capacity expansion project. Typically, the resulting objective function is accounted using the *expected net present value* or *ENPV*. In addition to the two-stage optimization, other approaches have been proposed to deal with uncertainties in the model parameters such as chance-constrained optimization (Charnes and Cooper, 1959), fuzzy programming (Bellman and Zadeh, 1970; Zimmermann, 1987), and the design flexibility approach (Ierapetritou and Pistikopoulos, 1994).

In the chance-constrained approach, some of the problem constraints are expressed in terms of probabilistic statements,

Correspondence concerning this article should be addressed to M. J. Bagajewicz at bagajewicz@ou.edu.

typically requiring that they be satisfied with a probability greater than a desired level. This approach is particularly useful when the cost and benefits of second-stage decisions are difficult to assess because the use of second-stage or recourse actions is avoided.

In turn, fuzzy programming assumes that the uncertain parameters can be represented by fuzzy numbers, whereas constraints are considered fuzzy sets. This approach also allows some constraint violation and their degree of satisfaction is defined as the membership function of the constraint. A comparison of fuzzy and two-stage stochastic programming made by Liu and Sahinidis (1996), in the context of capacity planning problems, showed that the latter offers several advantages over the former.

Finally, Ierapetritou and Pistikopoulos (1994) proposed an approach in which the plan or design is feasible only inside a certain region of the uncertainty space rather than for all possible uncertainty realizations. Then, a flexibility index is used to measure the extent of the plan's feasible uncertainty region. In the cited article, the authors used a flexibility index that represents the largest hypercube that can be inscribed inside the plan's feasible uncertainty region. In this approach, however, it is difficult to assess the trade-off between cost and flexibility.

A major limitation of all the mentioned approaches is that they consider, in one way or another, "expected outcomes" of the problem objective without explicitly taking into account its variability. Specifically, the two-stage stochastic models do not take into account the variability of the second-stage cost or profit but only its expected value. This was first discussed by Eppen et al (1989) in their work on automotive industry capacity planning. They proposed to use the concept of *downside risk* to measure the recourse cost variability and obtain solutions appealing to a risk-averse investor. Another approach to deal with the second-stage cost variability was proposed by Mulvey et al. (1995), who introduced the concept of *robustness* as the property of a solution for which the objective value for any realized scenario remains "close" to the expected objective value over all possible scenarios. Originally, the models for robust planning under uncertainty presented by Mulvey et al. (1995) used the variance of the cost as a "measure" of the robustness of the plan: that is, less variance corresponds to higher robustness. More recently, Ahmed and Sahinidis (1998), aiming at eliminating the nonlinearities introduced by the variance, proposed the use of the *upper partial mean* (*UPM*) as a measure of the variability of the recourse costs. They also offer a complete literature review on the problem. In addition to its linearity, the main advantage of using the *UPM*, as opposed to the variance, is its asymmetric nature that penalizes only the unfavorable cases from a risk perspective. However, the *UPM* suffers from limitations that make it an inappropriate measure to assess and manage financial risk. Because of the way the *UPM* is defined, a solution may falsely reduce its variability just by not choosing optimal second-stage decisions. Making nonoptimal second-stage decisions reduces the expected profit, allowing the positive deviation between the expected second-stage profit and the profit for that scenario (Δ_s) to be zero for some scenarios that otherwise would have a profit lower than the actual expected value and therefore Δ_s greater than zero. Because nobody would want to obtain a lower profit when a higher value is already attainable, operating with nonoptimal second-stage policies does not make sense from a financial point of view. This is discussed in detail by Takriti and Ahmed (2003), who present sufficient conditions for the variability measure of a robust optimization to ensure that the solutions are optimal in profit.

A different perspective to evaluate risk is also presented in the work of Ierapetritou and Pistikopoulos (1994). In that article, the authors proposed to use regret functions as an indirect measure of financial risk. For any realization of uncertain parameters, the regret function measures the difference between the objective function resulting from the actual plan or design, and the plan that is optimal for that realization of uncertain parameters. Then, the idea is to find plans that have low regret for the set of feasible uncertain parameters. This approach has two major difficulties. First, financial risk is evaluated only indirectly because the regret functions measure only the potential losses of the actual plan in comparison with a hypothetical plan that is optimal for only a specific uncertainty realization (no information is given about the feasibility of that plan under other circumstances). Thus, the regret functions do not provide any information about the financial risk. The second disadvantage of this approach is that to construct each regret function a separate optimization problem has to be solved for each possible uncertainty realization, which greatly increases the computational complexity of the problems.

Another approach, recently suggested by Cheng et al. (2003), is to rely on a Markov decision process modeling the design/production decisions at each epoch of the process as a two-stage stochastic program. The Markov decision process used is similar in nature to a multistage stochastic programming where structural decisions are also considered as possible recourse actions. Their solution procedure relies on dynamic programming techniques and is applicable only if the problems are separable and monotone. In addition, they propose to depart from single-objective paradigms, and use a multiobjective approach, rightfully claiming that cost is not necessarily the only objective and that other objectives are usually also important, like social consequences, environmental impact, and process sustainability, for example. Among these other objectives, they include risk (measured by downside risk, as introduced by Eppen et al., 1989), which under the assumption that decision makers are risk-averse, they claim should be minimized.

Aside from the fact that some level of risk could be tolerable at low profit aspirations to achieve larger gains at higher ones, thus promoting a risk-taking attitude, this assumption has some important additional limitations. As it will become apparent later in this article, given that downside risk is a function not only of the first-stage decisions but also of the aspiration or target profit level, minimizing downside risk at one level does not imply its minimization at another. Moreover, minimizing downside risk does not necessarily lead to minimizing financial risk for the specified target, a result that is discussed later in this article. Thus, treating financial risk as a single objective presents some limitations, and we propose that risk be managed over the entire range of aspiration levels. Applequist et al. (2000) proposed to manage risk at the design stages by using the concept of risk premium. They observed that for a variety of investments, the rate of return correlates linearly with the variability, which leads to the definition *risk premium*. Based on this observation, they suggest benchmarking new investments against the historical risk premium mark. Thus, they propose a two-objective problem, where the expected net present value and the risk premium are both maximized. The technique relies on using the variance as a measure of variability and therefore it penalizes scenarios at both sides of the mean equally, which is the same limitation discussed above. Recently, Gupta and Maranas (2003), while analyzing risk, also realized that symmetric measures were disadvantageous and therefore proposed to use a function very similar to the one we discuss in this article, but due to computational problems they resort to maximize the worst case scenario outcome. We point out that their definition of risk similar to the one proposed earlier by Barbaro and Bagajewicz (2003) have been used previously to assess (but not manage) risk, one of the most notorious examples being the petroleum exploration and production field (McCray, 1975).

The main objective of this article is to develop new mathematical formulations for problems dealing with planning and design under uncertainty that allow management of financial risk according to the decision maker's preference. A major step toward this objective is the use of a formal probabilistic definition of financial risk. In addition to this, the connection between downside risk, first introduced by Eppen et al. (1989), and financial risk is discussed. Using these two definitions, new two-stage stochastic programming models that are able to manage financial risk are developed. The advantages of the proposed approaches are that they maintain the original MILP structure of the problem. The theory developed in this article is of general application to any planning and design under uncertainty problem that can be formulated using a two-stage stochastic formulation.

The article is organized as follows. The "Two-Stage Stochastic Programming" section reviews general aspects of the two-stage stochastic formulation for planning under uncertainty. A theoretical definition of financial risk is introduced in the "Financial Risk Management" section, and the "Downside Risk: An Advantageous Measure to Assess and Manage Financial Risk" and "Other Measure of Risk: Value at Risk and Risk Adjusted Project Value" sections explore the connection between this definition and other risk measures. The "Two-Stage Stochastic Programming with Financial Risk Constraints" and "Two-Stage Stochastic Programming Using Downside Risk" sections outline new two-stage stochastic programming models to manage financial risk that are then applied to an illustrative process planning problem in the "Illustrative Example" section. Finally, "Computational Issues for Large-Scale Problems Using Model RO-SP-DR" discusses some issues related to the computational performance of the proposed formulations.

Two-Stage Stochastic Programming

This kind of optimization problems is characterized by two essential features: the uncertainty in the problem data and the sequence of decisions. Some of the model parameters are considered random variables with a certain probability distribution. In turn, some decisions are taken at the planning stage, that is, before the uncertainty is revealed, whereas a number of other decisions can be made only after the uncertain data become known. The first class of decisions is called the *first-stage decisions*, and the period when these decisions are taken is referred to as the first stage. On the other hand, the decisions made after the uncertainty is unveiled are called *second-stage* or *recourse decisions* and the corresponding period is called the second stage. Typically, first-stage decisions are structural and most of the time related to capital investment at the beginning of the project, whereas the second-stage decisions are often operational. Yet, some structural decisions corresponding to a future time can be considered as a second stage, that is, one may want to wait until some uncertainty (not necessarily all) is realized to make additional structural decisions. This kind of situations is formulated through the so-called *multistage models*, which are a natural extension of the two-stage case. Among the two-stage stochastic models, the expected value of the cost (or profit) resulting from optimally adapting the plan according to the realizations of uncertain parameters is referred to as the *recourse function*. Thus, a problem is said to have *complete recourse* if the recourse cost (or profit) for every possible uncertainty realization remains finite, independently of the nature of the first-stage decisions. In turn, if this statement is true only for the set of feasible first-stage decisions, the problem is said to have *relatively complete recourse* (Birge and Louveaux, 1997). This condition means that for every feasible first-stage decision, there is a way of adapting the plan to the realization of uncertain parameters. Another important property of certain two-stage problems, referred to as *fixed recourse*, will be discussed later. These properties are highly desirable and are found in most practical applications of this kind of optimization problems.

A large and useful collection of literature exists on two-stage stochastic programming modeling and solution techniques. Some excellent references are the books by Infanger (1994), Kall and Wallace (1994), Higle and Sen (1996), Birge and Louveaux (1997), Marti and Kall (1998), and Uryasev and Pardalos (2001). In addition, the articles by Pistikopoulos and Ierapetritou (1995), Cheung and Powell (1995), Iyer and Grossmann (1998), and Verweij et al. (2003) provide very good references on solution techniques for these problems.

The general extensive form of a two-stage mixed-integer linear stochastic problem for a finite number of scenarios can be written as follows (Birge and Louveaux, 1997).

Model SP

$$\text{Max } \mathrm{E}[Profit] = \sum_{s \in S} p_s q_s^T y_s - c^T x \tag{1}$$

s.t.

$$Ax = b \tag{2}$$

$$T_s x + W y_s = h_s \qquad \forall \ \ s \in S \tag{3}$$

$$x \geq 0 \quad x \in X \tag{4}$$

$$y_s \geq 0 \qquad \forall \ \ s \in S \tag{5}$$

In the above model, x represents the first-stage mixed-integer decision variables and y_s are the second-stage variables corresponding to scenario s, which has occurrence probability p_s. The objective function is composed of the expectation of the profit generated from operations minus the cost of first-stage

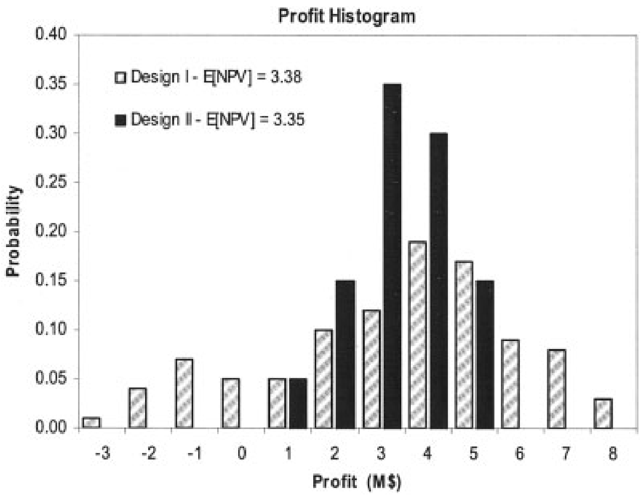

Figure 1. Solutions for the model SP with different financial risk levels.

decisions (capital investment). The uncertain parameters in this model appear in the coefficients q_s, the technology matrix T_s, and in the independent term h_s. In this article, the study was restricted to the cases where W, the recourse matrix, is deterministic. This is referred to in the literature as a problem with *fixed recourse* and ensures that the second-stage feasible region is convex and closed, and that the recourse function is a piecewise linear convex function in x (Birge and Louveaux, 1997). Cases where W is not fixed are found for instance in portfolio optimization when the interest rates are uncertain (Dupacova and Römisch, 1998). The above formulation considers the maximization of profit as an objective function but the same concepts and analysis developed in this article are valid for the case where the objective is the minimization of cost.

When trying to analyze the usefulness of model SP in the context of risk management one first notices that, even though it maximizes the total expected profit, it does not provide any control over the variability of the profit over the different scenarios. For instance, consider the profit histogram of two generic feasible solutions shown in Figure 1. The first design has a higher expected profit (M$ 3.38) than the second one (M$ 3.35); however, Design I is riskier than Design II because financial loss can occur under several scenarios. On the other hand, Design II renders positive profits for all scenarios.

Thus, a risk-averse investor would prefer Design II because it gives almost the same expected profit level and exhibits lower financial risk. This kind of preferences cannot be captured using model SP, because it does not contain any information about the variability of the profit. Then, a proper measure of financial risk needs to be included in the formulation to allow the decision maker to obtain solutions according to his/her desired risk exposure level.

Financial Risk Management

This section introduces several theoretical aspects for risk management. A formal definition of financial risk in the framework of two-stage stochastic programming is first introduced and then analyzed in terms of the profit probability distribution. This is the core of the risk management strategies that will be presented later in this article.

Probabilistic definition of financial risk

Financial risk associated with a planning project can be defined as the probability of not meeting a certain target profit (maximization) or cost (minimization) level referred to as Ω. For the two-stage stochastic problem (SP), the financial risk associated with a design x and a target profit Ω is therefore expressed by the following probability

$$Risk(x, \Omega) = P[Profit(x) < \Omega] \tag{6}$$

where $Profit(x)$ is the actual profit, that is, the profit resulting after the uncertainty has been unveiled and a scenario realized. As stated above, this definition has been made before (McCray, 1975). For instance, if one revisits the examples shown in Figure 1, one can see there is a 12% probability that Design I does not make a positive profit ($\Omega = 0$). Similarly, Design II has no risk of yielding negative profits, that is, $Risk$(Design II, 0) $= 0$.

To obtain an explicit expression for financial risk, let the profit corresponding to the realization of each scenario be

$$Profit_s(x) = q_s^T y_s - c^T x \qquad \forall\ s \in S \tag{7}$$

where y_s is the optimal second-stage solution for scenario s. Because uncertainty in the two-stage formulation is represented through a finite number of independent and mutually exclusive scenarios, the above probability can be expressed in terms of the probability of not meeting the target profit in each individual scenario realization

$$Risk(x, \Omega) = \sum_{s \in S} P[Profit_s(x) < \Omega] = \sum_{s \in S} P(q_s^T y_s - c^T x < \Omega) \tag{8}$$

Furthermore, for a given design the probability of not meeting the target profit in each particular scenario is either zero or one. That is, for any scenario, the profit is either greater or equal than the target level, in which case the correspondent probability $P[Profit_s(x) < \Omega]$ is zero, or the profit for the scenario is smaller than the target, rendering a probability of one. Therefore, the definition of risk can be rewritten as follows

$$Risk(x, \Omega) = \sum_{s \in S} p_s z_s(x, \Omega) \tag{9}$$

where z_s is a new binary variable defined for each scenario, as follows

$$z_s(x, \Omega) = \begin{cases} 1 & \text{If } q_s^T y_s - c^T x < \Omega \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ s \in S \tag{10}$$

Equations 9 and 10 constitute a formal definition of financial risk for two-stage stochastic problems with fixed recourse and discrete scenarios. This definition can now be

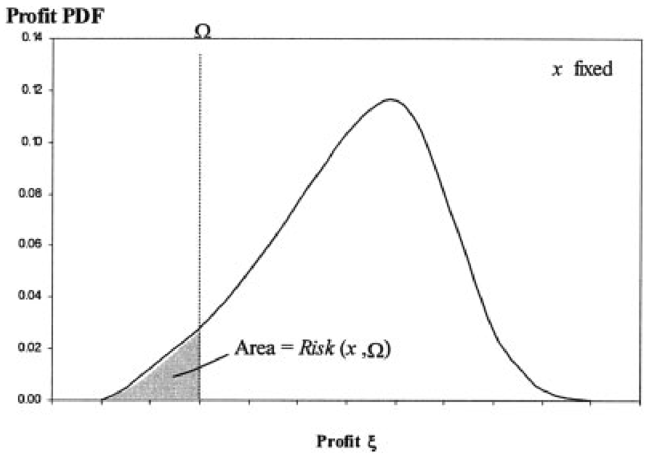

Figure 2. Probabilistic definition of financial risk: continuous case.

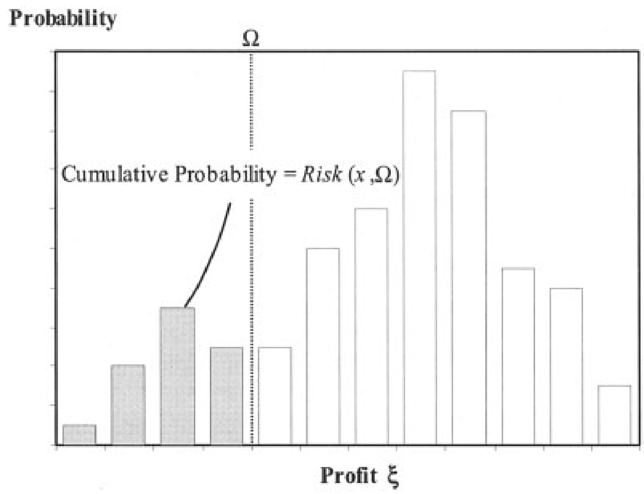

Figure 3. Probabilistic definition of financial risk: discrete case.

used to assess and manage the amount of risk related to the investment plan.

For conceptual purposes, the extension of this definition to the case where the uncertainty is represented by a continuous probabilistic distribution is now discussed. Intuitively, one may think of this case as a limiting one, where the number of scenarios becomes increasingly large, that is, $Cardinality(S) \to \infty$. Therefore, when profit has a continuous probability distribution, financial risk—defined as the probability of not meeting a target profit Ω—can be expressed as

$$Risk(x, \Omega) = \int_{-\infty}^{\Omega} f(x, \xi)d\xi \qquad (11)$$

where $f(x, \xi)$ is the profit probability distribution function (PDF), which is shown in Figure 2. The equivalent of the PDF in the discrete case is a *histogram* of frequencies similar to that depicted in Figure 1. A formal connection between the risk definition for the continuous case (Eq. 11) and the one for the discrete scenario-based case (Eq. 9) is provided in Appendix A.

From the integral in Eq. 11, it follows that financial risk associated to design x and a target profit Ω is given by the area under the curve $f(x, \xi)$ from $\xi = -\infty$ to $\xi = \Omega$, as shown in Figure 2. Alternatively, in the discrete scenario case, financial risk is given by the cumulative frequency obtained from the profit histogram as depicted by Figure 3.

A more straightforward way of assessing and understanding the trade-offs between risk and profit is to use the *cumulative risk curve*, as depicted in Figure 4.

When only a finite number of scenarios are considered, a discontinuous step-shaped cumulative risk curve is obtained. However, when the number of scenarios increases, the curve approaches continuous behavior, as shown in Figure 5.

For a given design x, the cumulative risk curve shows the level of incurred financial risk at each profit level. The cumulative risk curve is monotonically increasing because it is a cumulative probability function. Intuitively, one can see that the risk of not achieving relatively small profits will generally be small, whereas very high profit levels will exhibit large risk. Handling the shape and position of the curve is the main concern of the decision maker. A risk-averse investor may want to have low risk for some conservative profit aspiration level, whereas a risk-taker decision maker would prefer to see lower risk at higher profit aspiration level, even if the risk at lower profit values increases. Figure 6 illustrates a hypothetical example with two risk curves responding to these different risk attitudes.

Relationship between financial risk and expected profit

It seems obvious that there exists a direct relationship between risk and expected profit. Qualitatively, one would think that designs with large expected profits exhibit considerable risk, although having a quantitative relation is more beneficial. This relationship between financial risk and expected profit is discussed next. For a continuous probability case, the expected value of the profit can be written as

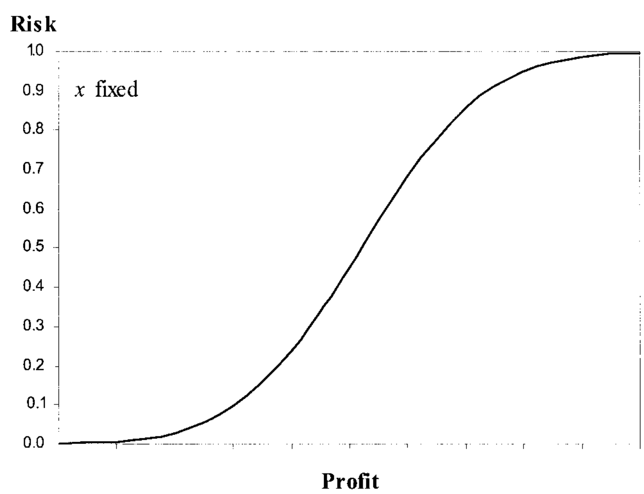

Figure 4. Characteristic behavior of the cumulative risk curve.

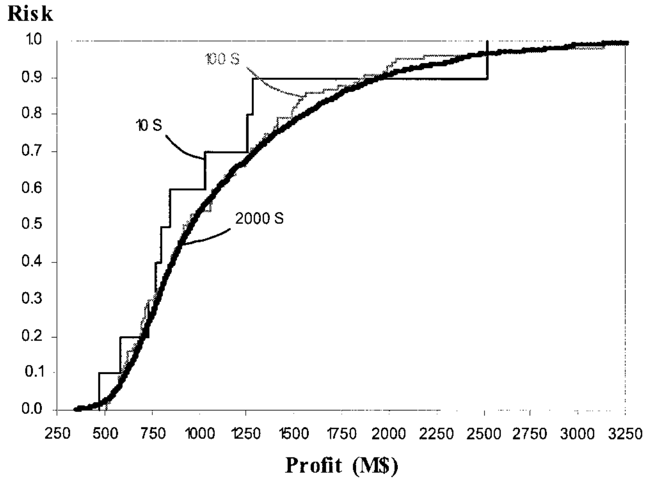

Figure 5. Risk curves obtained for different number of scenarios.

$$\mathrm{E}[Profit(x)] = \int_{-\infty}^{+\infty} \xi f(x, \xi) d\xi \qquad (12)$$

At this point, the analysis is focused in cases where $Profit(x)$ is a bounded random variable, that is, there exist finite real numbers $\underline{\xi}$ and $\bar{\xi}$, such that $\underline{\xi} < Profit(x) < \bar{\xi}$ for any realization of uncertain parameters. Under these circumstances, $Risk(x, \xi) = 1 \ \forall \ \xi \geq \bar{\xi}$. Then, using the definition of financial risk given by Eq. 11, the expected profit is

$$\mathrm{E}[Profit(x)] = \int_{0}^{1} \xi dRisk(x, \xi) \qquad (13)$$

Finally, integrating by parts one arrives at

$$\mathrm{E}[Profit(x)] = \bar{\xi} - \int_{\underline{\xi}}^{\bar{\xi}} Risk(x, \xi) d\xi \qquad (14)$$

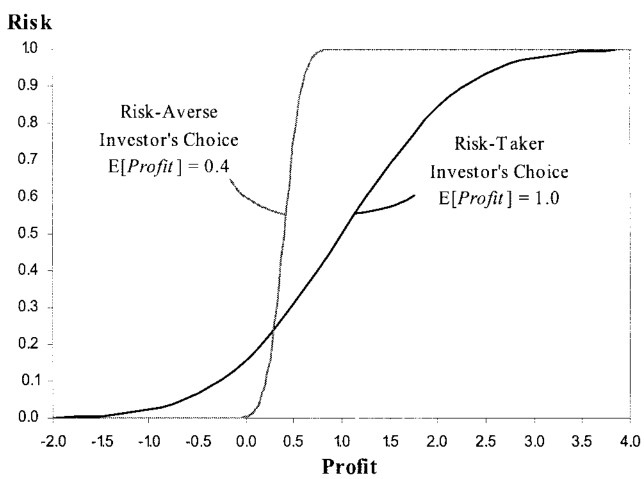

Figure 6. Different kinds of financial risk curves.

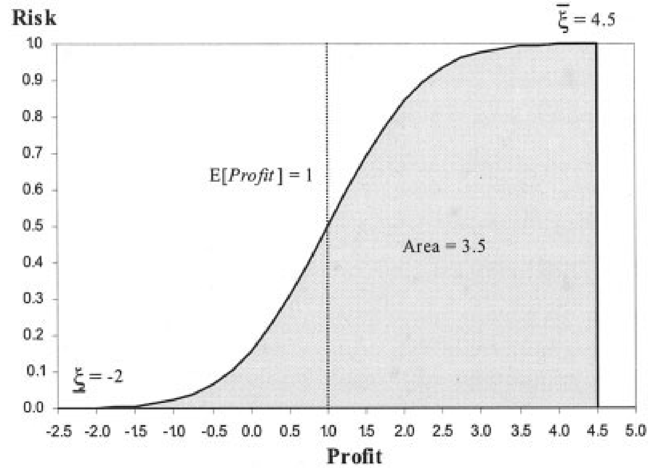

Figure 7. Relationship between expected profit and financial risk: continuous case.

Equation 14 is the quantitative relationship between expected profit and financial risk. Notice that a reduction in risk over the entire range of profits translates into a larger expected profit. In other words, this means that if $Risk(x, \xi)$ is minimized at every profit in the range $\underline{\xi} \leq \xi \leq \bar{\xi}$, then from Eq. 14 we obtain that $\mathrm{E}[Profit(x)]$ is maximized, given that $\bar{\xi}$ is just a constant. A graphical representation of this expression is provided by Figure 7.

Similarly, for a discrete scenario-based case the expected profit is given by

$$\mathrm{E}[Profit(x)] = \sum_{s \in S} p_s \xi_s \qquad (15)$$

After some manipulations, described in detail in Appendix B, the relationship between risk and expected profit is

$$\mathrm{E}[Profit(x)] = \bar{\xi} - \sum_{s \in S} Risk(x, \xi)(\xi_{s+1} - \xi_s) \qquad (16)$$

In this expression, the scenarios are sorted in ascending profit order, such that $\xi_{s+1} > \xi_s$. Therefore, the expected profit is also obtained by subtracting the area under the cumulative risk curve to the profit upper bound, as depicted in Figure 8.

Properties of financial risk curves

If risk is to be managed, one not only needs to measure it but also understand how risk curves behave as different designs are proposed. The following theorem, proven formally in Appendix C, states that for some profit level, any feasible solution to problem SP is riskier than the optimal solution.

Theorem 1 *Let x^* denote the optimal values of the first-stage variables for problem SP and x the values of first-stage variables for any other feasible solution with* $\mathrm{E}[Profit(x)] < \mathrm{E}[Profit(x^*)]$ and $\mathrm{E}[Profit(x^*)] < \infty$. *Then, there exists $\Omega \in \Re$ such that $Risk(x, \Omega) > Risk(x^*, \Omega)$.*

In other words, no feasible design x has a risk curve that lies entirely below the risk curve of the optimal solution to problem

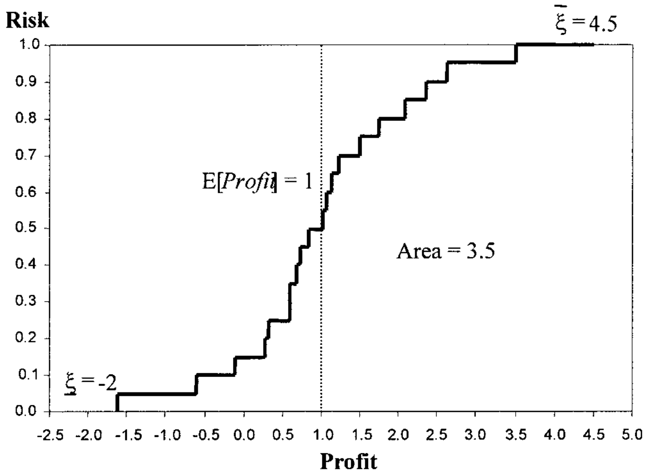

Figure 8. Relationship between expected profit and financial risk: discrete case.

SP and both risk curves either cross at some point(s) or the latter lies entirely above the former, as depicted in Figure 9.

Downside Risk: An Advantageous Measure to Assess and Manage Financial Risk

This section explores the relationship between financial risk, as defined earlier in this article, and another risk measure referred in the literature as *downside risk*. This measure was first introduced by Eppen et al. (1989) in the framework of capacity planning for the automobile industry. To present the concept of downside risk, let us first define $\delta(x, \Omega)$ as the positive deviation from a profit target Ω for design x, that is

$$\delta(x, \Omega) = \begin{cases} \Omega - Profit(x) & \text{If } Profit(x) < \Omega \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Downside risk is then defined as the expected value of $\delta(x, \Omega)$

$$DRisk(x, \Omega) = \mathrm{E}[\delta(x, \Omega)] \tag{18}$$

To incorporate the concept of downside risk in the framework of two-stage stochastic models let $\delta_s(x, \Omega)$ be the positive deviation from the profit target Ω for design x and scenario s defined as follows

$$\delta_s(x, \Omega) = \begin{cases} \Omega - Profit_s(x) & \text{If } Profit_s(x) < \Omega \\ 0 & \text{otherwise} \end{cases} \quad \forall\ s \in S \tag{19}$$

Because the scenarios are probabilistically independent, the expected value of $\delta(x, \Omega)$ (i.e., downside risk) can be expressed as the following linear function of δ

$$DRisk(x, \Omega) = \sum_{s \in S} p_s \delta_s(x, \Omega) \tag{20}$$

Similarly, in the case where the profit has a continuous probability distribution, downside risk is given by

$$DRisk(x, \Omega) = \int_{-\infty}^{\Omega} (\Omega - \xi) f(x, \xi) d\xi \tag{21}$$

Looking at the above definitions of downside risk, one must notice that downside risk is an expectation in \$, differing with the definition of $Risk(x, \Omega)$ that represents a probability value. Moreover, $DRisk(x, \Omega)$ is a continuous linear measure because it does not require the use of binary variables in the two-stage formulation because δ_s is a positive continuous variable. This is a highly desirable property to potentially reduce the computational requirements of the models to manage risk. Because of its linear and continue nature, using $DRisk(x, \Omega)$ instead of $Risk(x, \Omega)$ in the two-stage formulation framework allows the use of decomposition techniques developed for linear-continuous second-stage problems.

Now, a quantitative relationship $DRisk(x, \Omega)$ and $Risk(x, \Omega)$ is explored. In the continuous distribution case, separating the integral in Eq. 21 in two by distributing the product and remembering that $f(x, \xi)d\xi \equiv dRisk(x, \xi)$, one obtains

$$DRisk(x, \Omega) = \Omega \int_0^{Risk(x,\Omega)} dRisk(x, \xi) - \int_0^{Risk(x,\Omega)} \xi dRisk(x, \xi) \tag{22}$$

The first integral in the above equation is just $Risk(x, \Omega)$, and integrating by parts the second integral one obtains

$$DRisk(x, \Omega) = \Omega Risk(x, \xi) - \left[\Omega Risk(x, \xi) - \int_{-\infty}^{\Omega} Risk(x, \xi) d\xi\right] \tag{23}$$

Canceling out the first two terms in Eq. 23 provides the sought relationship between downside and financial risk

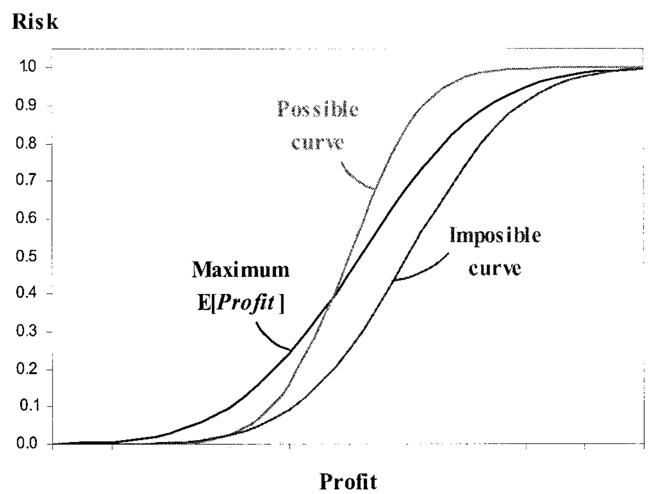

Figure 9. Possible risk curves.

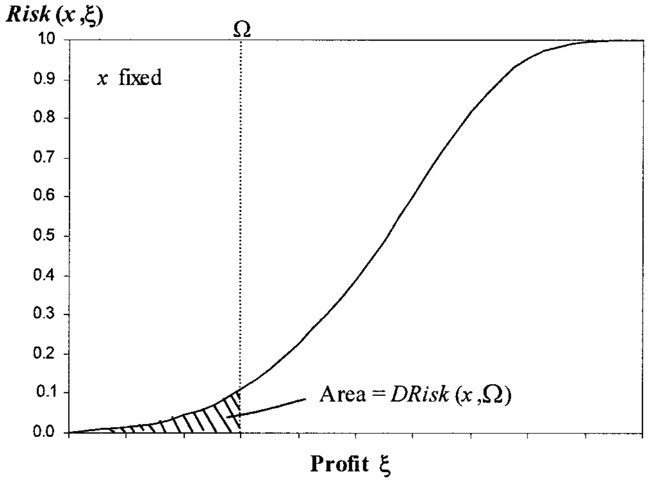

Figure 10. Interpretation of downside risk.

$$DRisk(x, \Omega) = \int_{-\infty}^{\Omega} Risk(x, \xi) d\xi \qquad (24)$$

Therefore, downside risk is defined as the integral of financial risk or area under the cumulative risk curve from profits $\xi = -\infty$ to $\xi = \Omega$, as shown in Figure 10.

In the previous section, Eq. 14 showed that the expected profit is related to the area under the cumulative risk curve from profits $\underline{\xi}$ to $\bar{\xi}$. Considering that downside risk is also defined as the area under the risk curve, we obtain a straightforward relationship between expected profit and downside risk

$$\mathrm{E}[Profit(x)] = \bar{\xi} - DRisk(x, \bar{\xi}) \qquad (25)$$

At this point, an interesting observation concerning the relationship between downside and financial risk can be drawn keeping in mind the decision maker's intention to minimize risk at every profit level. Because $Risk(x, \Omega)$ is an increasing function of the profit target, an intuitive way for reducing it over an entire continuous interval ($\underline{\Omega} \leq \Omega \leq \bar{\Omega}$) would be to minimize the area under it, that is, to minimize the value of the integral of $Risk(x, \Omega)$ from $\underline{\Omega}$ to $\bar{\Omega}$. This is the exact definition of downside risk as given in Eq. 24. Then, one may presume that minimizing $DRisk(x, \Omega)$ would be a convenient way of accomplishing the goal of minimizing financial risk. One must notice, however, that reducing the downside risk at a target profit Ω does not explicitly mean that financial risk is minimum at every single value of profit in the interval $-\infty < \xi \leq \Omega$. This is because at a given value of Ω there might exist some risk curves that having the same downside risk (that is, area) still exhibit different financial risk at some profits below Ω. A hypothetical example of this behavior is depicted in Figure 11 for a profit target $\Omega = 0.5$.

In view of the above discussion, one needs to analyze what should be the profit interval where minimizing downside risk helps to find solutions that satisfy the risk preference of the decision maker.

Other Measures of Risk: Value at Risk and Downside Expected Profit

Financial risk measured using the *Risk* function provides the probability of achieving any given aspiration level. In the risk management literature it is also usual to report the expected value of profit connected to such probability or confidence level. A widely used measure of this kind is referred as *Value at Risk* or *VaR* (Jorion, 2000), which was introduced by J. P. Morgan (Guldimann, 2000) and is defined as the expected loss for a certain confidence level, usually set at 5% (Linsmeier and Pearson, 2000). A more general definition of *VaR* is given by the difference between the mean value of the profit and the profit value corresponding to the *p*-quantile. For instance, a portfolio that has a normal profit distribution with zero mean and variance σ, *VaR* is given by $z_p\sigma$, where z_p is the number of standard deviations corresponding to the *p*-quantile of the profit distribution. Most of the uses of *VaR* are concentrated on applications where the profit probability distribution is assumed to follow a known distribution (usually the normal) so that it can be calculated analytically. The relationship between *VaR* and *Risk* is generalized as follows

$$VaR(x, p_\Omega) = \mathrm{E}[Profit(x)] - Risk^{-1}(x, \Omega) \qquad (26)$$

where p_Ω is the confidence level related to profit Ω, that is, $p_\Omega = Risk(x, \Omega)$. Notice that *VaR* requires the computation of the inverse function of *Risk*. Moreover, because *Risk* is a monotonically increasing function of Ω, one can see from Eq. 26 that *VaR* is a monotonically decreasing function of p_Ω.

For two-stage stochastic problems with a finite number of scenarios, *VaR* can be easily estimated by sorting the scenarios in ascending profit order and simply taking the profit value of the scenario for which the cumulative probability equals the specified confidence level; that is

$$VaR(x, p_\Omega) = \mathrm{E}[Profit(x)] - Profit_{s_p}(x)$$

$$s_p = \left\{ s \,\middle|\, \sum_{k=1}^{s} p_k = p_\Omega \right\} \qquad (27)$$

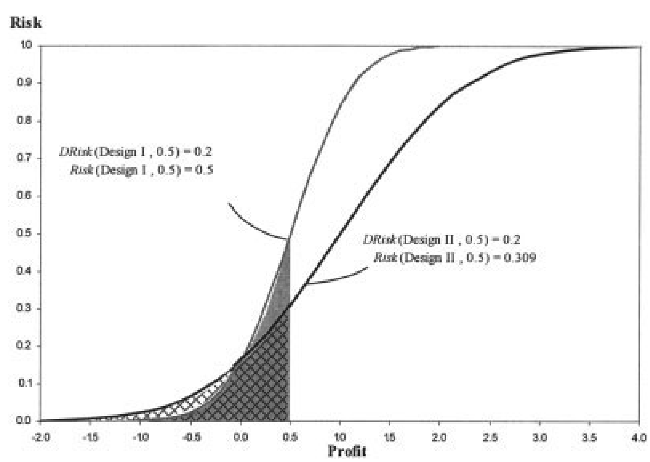

Figure 11. Downside and financial risk.

Thus, calculating *VaR* as a postoptimization measure of risk is a simple task and does not require any assumptions on the profit distribution. The difficulty rises when one wants to include *VaR* as a measure of risk inside the optimization, whether in the objective or as a restrictive constraint, because even though it could be handled through linear expressions, it would require many additional constraints. This is attributed to the fact that one would basically need to construct the logic for the above describing scenario sorting by means of linear constraints. Given this computational shortcoming and the fact that *VaR* is closely related to *Risk*, we conclude that it is more convenient to use *VaR* as a risk indicator after the optimization is performed rather than a measure used within the optimization.

Another measure of risk that one can propose is the *Downside Expected Profit* (*DEP*) for a confidence level p_Ω, defined formally as follows

$$DEP(x, p_\Omega) = \int_{-\infty}^{\Omega} \xi f(x, \xi) d\xi$$

$$= \Omega Risk(x, \Omega) - dRisk(x, \Omega) \quad (28)$$

Notice that *DEP* is a monotonically increasing function of p_Ω. Because the profit distributions are not symmetric and usually have different variance, plotting *DEP* as a function of the risk can be revealing because at low risk values some feasible solutions may exhibit larger expected profit. This measure is illustrated later in an example. Another related risk adjusted measure is the Risk Adjusted Return on Capital (*RAROC*), which is the quotient of the difference between the expected profit of the project adjusted by risk and the capital (or value) at risk of an equivalent investment and the value at risk. This value is known to be a multiple of the Sharpe ratio in portfolio optimization. The intricacies of this relationship and its possible use in two-stage stochastic models to manage risk are left for future work.

Two-Stage Stochastic Programming with Financial Risk Constraints

This section introduces a new mathematical formulation to assess and manage financial risk. The idea behind this formulation is that the decision maker wants to maximize the expected profit and at the same time minimize the financial risk at every profit level. At first sight this may appear as a two-objective trade-off; however, it is interesting to note from the relationship derived in Eq. 14 that a solution that minimizes financial risk at *every* profit target also maximizes the expected net present profit. We emphasize here the qualifier word *every*, given that this statement is not true if only a subset of profit targets is considered. Thus, minimizing $Risk(x, \Omega)\ \forall\ \Omega \in \Re$ and maximizing $E[Profit(x)]$ are equivalent objectives. However, minimization of risk at some profit levels renders a trade-off with expected profit. As discussed before, a risk-averse decision maker will feel more comfortable with low risk at low values of Ω, whereas a risk taker will prefer to lower the risk at high values of Ω. The trade-off lies in the fact that minimizing risk at low values of Ω (such as a loss) is in conflict with the minimization of risk at high values of Ω (such as large profits) and vice versa.

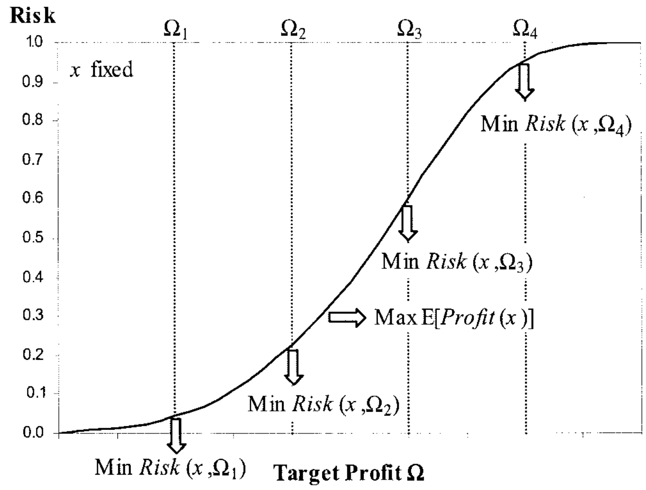

Figure 12. Multiobjective approach.

From a mathematical programming point of view, minimizing $Risk(x, \Omega)$ for a continuous range of profit targets Ω results in an infinite multiobjective optimization problem. Even though this model would be able to reflect the decision maker's intention, having an infinite optimization problem would be computationally prohibitive. However, one can approximate the ideal infinite optimization approach by a finite multiobjective problem that only minimizes risk at some finite number of profit targets and maximizes the expected profit, as shown in Figure 12.

In this case, the objective of maximizing expected profit should be included because risk is minimized at some values and not for the entire continuous range. Additionally, including the expected profit as objective will tend to reduce the risk at every profit because of the relationship given by Eq. 14. The finite multiobjective formulation is detailed next.

Finite multiobjective formulation

$$\text{Max } E[Profit] = \sum_s p_s q_s^T y_s - c^T x \quad (29)$$

$$\text{Min } Risk(\Omega_1) = \sum_s p_s z_{s1}$$

$$\vdots$$

$$\text{Min } Risk(\Omega_i) = \sum_s p_s z_{si}$$

s.t.

Constraints 2 to 5

$$q_s^T y_s - c^T x \geq \Omega_i - U_s z_{si} \qquad \forall\ s \in S, i \in I \quad (30)$$

$$q_s^T y_s - c^T x \leq \Omega_i + U_s(1 - z_{si}) \qquad \forall\ s \in S, i \in I \quad (31)$$

$$z_{si} \in \{0, 1\} \qquad \forall\ s \in S, i \in I \quad (32)$$

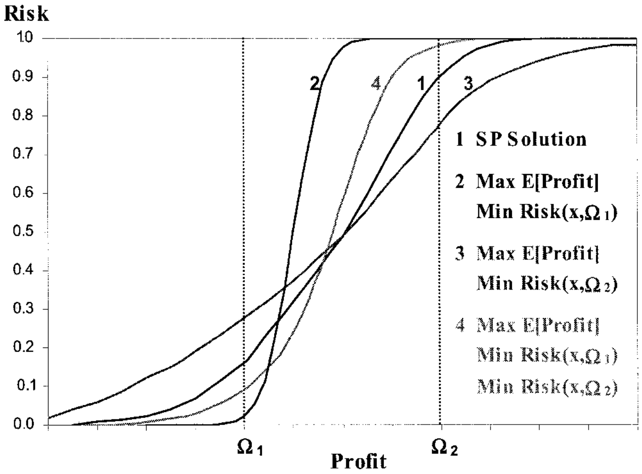

Figure 13. Spectrum of solutions obtained using a multiobjective approach.

In the above formulation, constraints 30 and 31 force the new integer variable z_{si} to take a value of zero if the profit for scenario s is greater than or equal to the target level (Ω_i) and a value of one otherwise. To do this, an upper bound of the profit of each scenario (U_s) is used. The value of the binary variables is then used to compute and penalize financial risk in the objective function.

To illustrate the usefulness of the above multiobjective formulation, consider a set of hypothetical solutions, as depicted in Figure 13, where Model SP was first solved to obtain the solution that maximizes the expected profit and two profit targets Ω_1 and Ω_2 were later used in the above multiobjective model to manage financial risk.

In Figure 13, Solutions 2 and 3 maximize the expected profit with minimum financial risk at profit targets Ω_1 and Ω_2, respectively. Thus, minimizing financial risk at each profit target independently of others targets will typically result in designs that perform well around the specific target but do poorly in the rest of the profit range. When risk, on the other hand, is minimized for every target at the same time, solutions that perform well in the entire range of interest may be found. Thus, using a multiobjective approach all of the solutions 1, 2, 3, and 4 can be obtained giving different importance to each of the objectives and, finally, it is up to the decision maker to make a choice accordingly with his/her risk preference. This is an important advantage over other approaches that are not able to provide a full spectrum of solutions. Two models and an algorithm to implement this multiobjective approach are introduced next. These models are different multiparametric representations of the above multiobjective model. The first one includes a goal programming weight $\rho_i \geq 0$ in the objective function to obtain solutions where the relative importance of expectation and risk is progressively changed, controlling the shape of the risk curve. This is done by imposing a penalty for risk at different target profits (Ω_i), as presented next.

Model RO-SP-FR

$$\text{Max} \sum_{s \in S} p_s q_s^T y_s - c^T x - \sum_{s \in S} \sum_{i \in I} p_s \rho_i z_{si} \tag{33}$$

s.t.

Constraints 2 to 5

Constraints 30 to 32

The second model to manage risk uses a restricted recourse approach where a new constraint imposes an upper bound to $Risk(x, \Omega_i)$, with the objective function remains the same as in model SP. This formulation is presented next.

Model RR-SP-FR

$$\text{Max} \sum_{s \in S} p_s q_s^T y_s - c^T x \tag{34}$$

s.t.

Constraints 2 to 5

Constraints 30 to 32

$$\sum_{s \in S} p_s z_{si} \leq \varepsilon_i \qquad \forall \ i \in I \tag{35}$$

In the above models the set I is corresponds to all the targeted profits. Given that the number of binary variables and the size of these formulations increase with the number of profit targets one should evaluate for each specific problem what are the benefits of considering multiple profit targets. In some cases one may find that setting only one profit target is sufficient to achieve the goal of obtaining solutions that match the decision maker's preference, as will be shown in later in an illustrative example.

Observing the structure of the two multiparametric models presented above, one realizes that model RO-SP-FR has the highly desirable property of separability, that is, there are no constraints linking different scenarios. On the contrary, formulation RR-SP-FR is nonseparable due to constraint 35, which is a first-stage constraint since the summation runs over all possible scenarios. This means that model RO-SP-FR should be computationally more efficient than model RR-SP-FR because many techniques have been developed for solving separable two-stage stochastic problems efficiently (Birge and Louveaux, 1997). Thus, if both formulations were equivalent, meaning that they have the same optimal solution for appropriate choices of ρ and ε, then formulation RO-SP-FR would be preferable. As stated in Theorem 2 (Appendix D) it turns out that these two formulations are equivalent.

Proving the equivalence between both formulations also allows proving that their optimal solutions are not stochastically dominated by any other solutions. A formal definition for stochastic dominance is the following: a solution **I** is stochastically dominated by other solution **II** if for every scenario the profit of solution **II** is at least as large as the correspondent to solution **I** and strictly greater for at least one scenario. This concept is also known in the multiobjective literature as *Pareto Optimality*. Hence, a solution is said to be *Pareto-optimal* if it is not stochastically dominated by any other solution. Clearly, if solution **I** is stochastically dominated by solution **II** then the expected profit of the latter is strictly greater than the one of the former. The Pareto optimality of the solutions is a desirable property because it guaranties that solutions with the maximum expectation possible are obtained. Theorem 3 in Appendix E proves that the solutions of both of the risk management models presented above are always Pareto-optimal. Taking a closer look at models RO-SP-FR and RR-SP-FR, one realizes that constraints 30 and 31 break the original fixed-recourse property of the problem because in these constraints the coefficients of the second-stage variables y_s are stochastic. Given that having a fixed recourse matrix ensures the feasible region of the second-stage problem is convex and closed (Birge and Louveaux, 1997), breaking this property seems at first sight to be a considerable disadvantage of the proposed formulations. However, Theorem 4 in Appendix F proves that the feasible region of the model RO-SP-FR is the same as the feasible region of model SP and therefore it is still convex and closed. This is also true for model RR-SP-FR, given that the values of ε_i make this model equivalent to model RO-SP-FR.

Next, an algorithm for the implementation of the multiobjective approach to financial risk management is presented. In view of the advantages mentioned in the previous discussion, model RO-SP-FR is used in this procedure.

Risk Management Procedure Using the Multiobjective Model RO-SP-FR

(1) Solve the SP problem to obtain a solution that maximizes the expected profit.

(2) Construct the corresponding risk curve. If the decision maker is satisfied with the current level of risk, then stop; otherwise, go to Step 3.

(3) Let the decision maker choose an arbitrary set of profit targets Ω_i for which financial risk is to be reduced, if possible. Additionally, for every target define a sequence of k_i weights $\rho_i^{k_i} = \{\rho_i^1, \ldots, \rho_i^{k_i}\}$ to manage the trade-off between expected profit and risk.

(4) Generate a set of $n = \Pi_i k_i$ instances of problem RO-SP-FR corresponding to all combinations of weights ρ for the different profit targets. Solve all instances and construct the resulting risk curves associated with the optimal solution of each instance.

(5) Let the decision maker evaluate the results. If the decision maker is satisfied with the risk curve of one or more solutions, then stop; otherwise, go to Step 3.

In summary, the formulations presented in this section address the problem of controlling the financial risk curve of the solutions to the two-stage stochastic problem such that the decision maker can satisfy his/her risk preference. Computationally, however, the inclusion of new integer variables could in some cases represent a major limitation of these formulations. Adding integer second-stage variables eliminates the possibility of using efficient decomposition techniques developed for linear problems with continuous variables. Approaches to solve stochastic programs with mixed-integer first- and second-stage variables were introduced by Caroe and Schultz (1997) and by Ahmed et al. (2000). However, the computational expense to solve large-scale problems using these methods is still significant. In additions, using Generalized Bender's Decomposition to solve these models can guarantee optimality only when the integer variables z_{si} are treated as first-stage decisions because nonconvex second-stage subproblems would arise otherwise. This limits the efficiency of the method for problems with large number of scenarios and first-stage decision variables that include integers. Thus, more research effort should be directed to efficiently solve these mixed-integer stochastic formulations. Nonetheless, for some applications the computational limitations can be overcome using presampling methods, such as the stochastic average approximation (Verweij et al., 2001), which yield modest size MILP instances. To ameliorate some of these problems an alternative measure of risk is discussed next.

Two-Stage Stochastic Programming Using Downside Risk

In this section we suggest that $DRisk(x, \Omega)$ be the measure used to control financial risk at different profit targets. In addition, Eq. 14 indicates that $E[Profit(x)]$ may also be taken as a measure of financial risk at higher profit levels. Including the expected profit in the objective would also help the model choose the solution with higher expectation in cases where two or more solutions exhibit the same downside risk at a target Ω. The proposed model is as follows.

Model RO-SP-DR

$$\text{Max } \mu\left(\sum_{s\in S} p_s q_s^T y_s - c^T x\right) - \sum_{s\in S} p_s \delta_s \tag{36}$$

s.t.

Constraints 2 to 5

$$\delta_s \geq \Omega + c^T x - q_s^T y_s \qquad \forall \ s \in S \tag{37}$$

$$\delta_s \geq 0 \qquad \forall \ s \in S \tag{38}$$

In this case also a procedure that generates a full spectrum of solutions is presented. This is accomplished by varying the profit target Ω from small values around $\Omega = \min_s \{Profit_s(x^*_{SP})\}$ up to higher values around $\Omega = \max_s \{Profit_s(x^*_{SP})\}$. In this way, solutions obtained for lower values of Ω will generally respond to a risk-averse investor and solutions obtained with higher Ω will be more appealing to risk-taker investors.

Parametric algorithm for financial risk management using downside risk

(1) Solve the SP problem to obtain a solution that maximizes the expected profit.

(2) Construct the corresponding risk curve. If the decision maker is satisfied with the current level of risk, then stop; otherwise, let $k = 1$ and go to Step 3.

(3) Define $\Omega^0 = \min_s\{Profit_s(x^*_{SP})\}$ and $\mu = 0.001$ (or an equivalent small number).

(4) Choose a profit target $\Omega^k > \Omega^{k-1}$. Let $l_k = 1$.

(5) Generate and solve problem RO-SP-DR using Ω^k. Add the new solution to the cumulative risk curve chart.

(6) Let the decision maker evaluate the results. If the decision maker is satisfied with the risk curve of one or more solutions, then stop; otherwise, let $k = k + 1$ and go to Step 4.

Quite clearly, different values of μ can possibly lead to different solution, even for the same profit target. The same is true for different values of $\rho_i^{k_i}$ in the case of the the RO-SP-FR model. Models with smaller value of μ will likely provide smaller *ENPV*. This is clearly another tool that the decision maker has to obtain more risk curves to choose from, as we shall see below.

Illustrative Example

This section illustrates the usefulness of the proposed risk management models using a capacity-planning problem. Consider the two-stage stochastic model presented by Liu and Sahinidis (1996), which is an extension of the deterministic mixed-integer linear programming formulation introduced by Sahinidis et al. (1989).

Model PP

$$\text{Max } ENPV = \sum_{s=1}^{NS}\sum_{t=1}^{NT} p_s L_t \left[\sum_{l=1}^{NM}\sum_{j=1}^{NC} (\gamma_{jlts}S_{jlts} - \Gamma_{jlts}P_{jlts}) - \sum_{i=1}^{NP} \delta_{its}W_{its}\right] - \sum_{i=1}^{NP}\sum_{t=1}^{NT} (\alpha_{it}E_{it} + \beta_{it}Y_{it}) \quad (39)$$

s.t.

$$Y_{it}E_{it}^L \le E_{it}^L \le Y_{it}E_{it}^U \qquad i = 1, \ldots, NP \quad t = 1, \ldots, NT \quad (40)$$

$$Q_{it} = Q_{i(t-1)} + E_{it} \qquad i = 1, \ldots, NP \quad t = 1, \ldots, NT \quad (41)$$

$$\sum_{t=1}^{NT} Y_{it} \le NEXP_i \qquad i = 1, \ldots, NP \quad (42)$$

$$\sum_{i=1}^{NP} (\alpha_{it}E_{it} + \beta_{it}Y_{it}) \le CI_t \qquad t = 1, \ldots, NT \quad (43)$$

$$W_{its} \le Q_{it} \qquad i = 1, \ldots, NP \quad t = 1, \ldots, NT \quad s = 1, \ldots, NS \quad (44)$$

$$\sum_{l=1}^{NM} P_{jlts} + \sum_{i=1}^{NP} \eta_{ij}W_{its} = \sum_{l=1}^{NM} S_{jlts} + \sum_{i=1}^{NP} \mu_{ij}W_{its} \qquad j = 1, \ldots, NC \quad t = 1, \ldots, NT \quad s = 1, \ldots, NS \quad (45)$$

$$a_{jlts}^L \le P_{jlts} \le a_{jlts}^U \qquad j = 1, \ldots, NC \quad l = 1, \ldots, NM \quad t = 1, \ldots, NT \quad s = 1, \ldots, NS \quad (46)$$

$$d_{jlts}^L \le S_{jlts} \le d_{jlts}^U \qquad j = 1, \ldots, NC \quad l = 1, \ldots, NM \quad t = 1, \ldots, NT \quad s = 1, \ldots, NS \quad (47)$$

$$Y_{it} \in \{0, 1\} \qquad i = 1, \ldots, NP \quad t = 1, \ldots, NT \quad (48)$$

$$E_{it}, Q_{it}, W_{its}, P_{jlts}, S_{jlts} \ge 0 \qquad \forall\ i, j, l, t, s \quad (49)$$

In this model, the objective function 39 maximizes the expected net present value (*ENPV*) over the two stages of the capacity expansion project, defined as the difference between sales revenues and the investment, operating and raw material costs. The investment cost in terms of the design variables is represented by a variable term that is proportional to the capacity expansion E and a fixed-charge term that taken into account using binary decision variables Y. The second stage or recourse cost is described as the expectation of the sales revenues and the expectation of the second-stage operating costs over finitely many, mutually exclusive scenarios s for each time period. Constraint 40 enforces lower and upper bounds in the capacity expansion by means of the binary variables Y. Constraint 41 defines the total capacity available for process i during time period t. Limits on the number of expansions of processes and the capital budget are imposed by inequalities 42 and 43, respectively. Constraint 44 ensures that the operating level of a process does not exceed the installed capacity. In turn, Eq. 45 expresses the material balances for each process, whereas constraints 46 and 47 enforce lower and upper bounds for raw materials availability and products sales on each market. Observe that constraints 40 through 43 are expressed only in terms of first-stage variables; hence they are referred as first-stage constraints. On the other hand, constraints 44 through 47 are indexed over all possible scenarios and are then called second-stage constraints. In the above formulation the different scenarios between different time periods have been treated independently because there are no second-stage constraints linking time periods. Thus, a total of *NS* independent scenarios per time period are considered. Next, two models to manage financial risk are presented, the first one using *Risk*(x, Ω) and the second one *DRisk*(x, Ω) as measures of risk.

Model RO-PP-FR

$$\text{Max} \sum_{s=1}^{NS}\sum_{t=1}^{NT} p_s L_s \left[\sum_{l=1}^{NM}\sum_{j=1}^{NC} (\gamma_{jlts}S_{jlts} - \Gamma_{jlts}P_{jlts}) - \sum_{i=1}^{NP} \delta_{its}W_{its}\right] - \sum_{i=1}^{NP}\sum_{t=1}^{NT} (\alpha_{it}E_{it} + \beta_{it}Y_{it}) - \sum_{s=1}^{NS}\sum_{n=1}^{NR} \rho_n p_s z_{sn} \quad (50)$$

s.t.

Constraints 40 to 49

$$\sum_{t=1}^{NT} L_t \left[\sum_{l=1}^{NM}\sum_{j=1}^{NC} (\gamma_{jlts}S_{jlts} - \Gamma_{jlts}P_{jlts}) - \sum_{i=1}^{NP} \delta_{its}W_{its}\right] - \sum_{i=1}^{NP}\sum_{t=1}^{NT} (\alpha_{it}E_{it} + \beta_{it}Y_{it}) \le \Omega_n + U_s(1 - z_{sn}) \qquad n = 1, \ldots, NR \quad s = 1, \ldots, NS \quad (51)$$

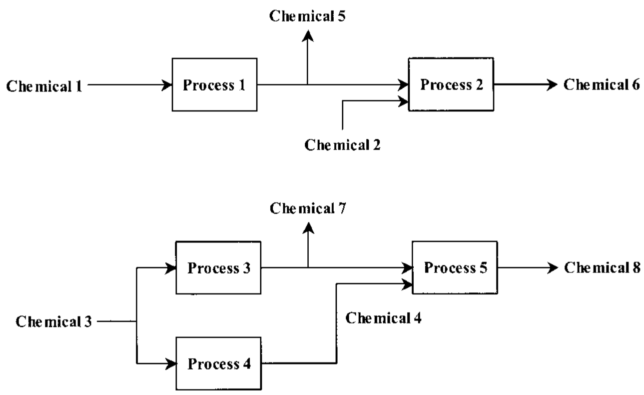

Figure 14. Process network for the example.

$$\sum_{t=1}^{NT} L_t\left[\sum_{l=1}^{NM}\sum_{j=1}^{NC}(\gamma_{jlts}S_{jlts} - \Gamma_{jlts}P_{jlts}) - \sum_{i=1}^{NP}\delta_{its}W_{its}\right] - \sum_{i=1}^{NP}\sum_{t=1}^{NT}(\alpha_{it}E_{it} + \beta_{it}Y_{it}) \geq \Omega_n - U_s z_{sn} \quad \begin{array}{l} n = 1, \ldots, NR \\ s = 1, \ldots, NS \end{array} \tag{52}$$

$$z_{sn} \in \{0, 1\} \quad \begin{array}{l} n = 1, \ldots, NR \\ s = 1, \ldots, NS \end{array} \tag{53}$$

Model RO-PP-DR

$$\text{Max } \mu\left\{\sum_{s=1}^{NS}\sum_{t=1}^{NT} p_s L_t\left[\sum_{l=1}^{NM}\sum_{j=1}^{NC}(\gamma_{jlts}S_{jlts} - \Gamma_{jlts}P_{jlts}) - \sum_{i=1}^{NP}\delta_{its}W_{its}\right] - \sum_{i=1}^{NP}\sum_{t=1}^{NT}(\alpha_{it}E_{it} + \beta_{it}Y_{it})\right\} - \sum_{s=1}^{NS} p_s\delta_s \tag{54}$$

s.t.

Constraints 40 to 49

$$\delta_s \geq \Omega + \sum_{i=1}^{NP}\sum_{t=1}^{NT}(\alpha_{it}E_{it} + \beta_{it}Y_{it}) - \sum_{t=1}^{NT} L_t\left[\sum_{l=1}^{NM}\sum_{j=1}^{NC}(\gamma_{jlts}S_{jlts} - \Gamma_{jlts}P_{jlts}) - \sum_{i=1}^{NP}\delta_{its}W_{its}\right] \quad s = 1, \ldots, NS \tag{55}$$

$$\delta_s \geq 0 \qquad s = 1, \ldots, NS \tag{56}$$

Table 1. Fixed Investment Cost Coefficients

Process	β_{it} (k\$)		
	t_1	t_2	t_3
i_1	20,000	19,000	18,000
i_2	21,000	19,000	17,000
i_3	40,000	39,500	39,000
i_4	44,000	42,500	40,000
i_5	48,000	46,000	44,000

Table 2. Variable Investment Cost Coefficients

Process	α_{it} (k\$ · yr/kton)		
	t_1	t_2	t_3
i_1	800	795	790
i_2	780	770	760
i_3	1400	1360	1320
i_4	1360	1340	1320
i_5	1300	1290	1280

Example

This example consists of an investment project involving five chemical processes and eight chemicals arranged in a process network, as shown in Figure 14. The project is staged in three periods of length one, two and a half, and three and a half years, respectively. The maximum number of expansions allowed for each process is two and the capital limits at each period are 100, 150, and 200 M\$, respectively. The upper bound on capacity expansion is 100 kton/yr for all processes in all periods and none of the processes had initial capacity installed. Tables 1 and 2 show the fixed and variable investment costs for each process, respectively, and Table 3 gives the operational costs. Mass balance coefficients for products and raw materials are given in Tables 4 and 5. All these parameters are considered deterministic (i.e., nonstochastic). On the other hand, market prices, availabilities, and demands were considered as uncertain parameters characterized by a normal probability distribution, with mean value and standard deviation as given in Tables 6 to 9. The uncertainty realizations for this problem were simulated through 400 independent scenarios generated by random sampling from the probability distributions of the problem parameters.

To manage financial risk for the above-described problem, model PP was solved first, obtaining the solution that maximizes the expected net present value, without taking financial risk into account. This solution, the risk curve for which is shown in Figure 15, is detailed in Table 10 and graphically represented in Figure 16.

After solving model PP the idea is then to explore the risk behavior of other solutions by using models RO-PP-FR and RO-PP-DR. In this way, the decision maker is provided a series of solutions reflecting different levels of risk exposure to make a selection according to his/her criteria. First, the results using model RO-PP-FR are presented.

Results using model RO-PP-FR

To explore different alternatives using this model, financial risk at several *NPV* targets from 600 to 1500 M\$ was minimized (with weight $\rho = 10{,}000$) considering one target at a time. The results for each target are shown in Table 11, where

Table 3. Operation Cost Coefficients

Process	δ_{it} (k\$ · yr/kton)		
	t_1	t_2	t_3
i_1	2000	2200	2400
i_2	2100	2000	1900
i_3	1500	1400	1300
i_4	1300	1300	1300
i_5	1400	1200	1100

Table 4. Raw Material Stoichiometric Coefficients

Process	μ_{ij} j_1	j_2	j_3	j_4	j_5	j_6	j_7	j_8
i_1					1			
i_2						1		
i_3							1	
i_4				1				
i_5								1

Table 6. Expected Purchase Prices and Availabilities

Market l_1 Chemical	Price (k$/kton) t_1	t_2	t_3	Availability (kton/yr) t_1	t_2	t_3
j_1	4000	4300	4600	50	60	70
j_2	5000	5500	6000	60	80	100
j_3	6000	6100	6200	40	42	44

the solution for model PP is also included for comparison purposes. The risk curves corresponding to the different solutions are shown in Figure 17. A total of 200 scenarios was used in all instances of problem RO-PP-FR.

Observe that for this example, solutions obtained by minimizing financial risk at targets below the maximum expected net present value of M$ 1140 exhibit lower risk of realizing a small *NPV*. On the other hand, when risk was minimized at higher *NPV* targets the solutions showed considerably higher risk at small profits and only a small reduction in risk at higher *NPV* values with respect to the solution given by model PP. Thus, minimizing risk at large *NPV* targets brings about higher risk at lower *NPV* values. This is in agreement with the theoretical behavior described in previous sections. Then, one can conclude that for this example it does not seem worthwhile to choose solutions that minimize risk at *NPV* targets higher than M$ 1000.

To further analyze the results, notice from Table 12 that as the *NPV* target (Ω) is incremented, the expected net present value of the solutions approaches to the maximum value obtained with model PP. This is attributed to the relationship between financial risk and expected profit established by Eq. 14, which shows that as the profit target increases and financial risk approaches unity, minimizing risk becomes equivalent to maximizing the expected profit.

So far, a set of solutions showing different risk characteristics was obtained. This set should next be analyzed by the decision maker and some solution(s) selected. In a situation where none of the solutions satisfies the decision maker's preference, more solutions should be explored using different targets Ω and weights ρ as described in "Two-Stage Stochastic Programming with Financial Risk Constraints." The procedure should be repeated until the decision maker is satisfied or financial risk cannot be managed further. One can conclude from Figure 17 that most of the risk behavior for this example has been already captured by the shown solutions.

One final observation can be made regarding the risk behavior of the decision makers: it is usually assumed that decision makers are risk-averse, that is, they want to lower their exposure to losses. However, this characteristic may be problem-dependent: for instance, in this example all solutions guarantee profit under every scenario. Moreover, solutions reducing risk to almost zero at a target of $\Omega = 600$ M$ also exhibit a probability greater than 50% of not making a profit larger than M$ 1000. On the other hand, the solution obtained without penalizing risk (using model SP), has only around 10% risk at this profit target, but a respectable chance of making much higher profits. Therefore, even a risk-averse decision maker may become a risk taker under these circumstances. This reasoning shows the importance of obtaining a full spectrum of solutions such as the one generated using the proposed models, and the need to present them to the decision maker so he/she can make the final choice.

Results using model RO-PP-DR

This section presents the results obtained with model RO-PP-DR. The strategy in this case was to handle the risk curves by minimizing the downside risk at several *NPV* targets from 600 to 1500 M$ with weight $\mu = 0.001$ for the expected *NPV*. The results for each target are shown in Table 12 and the corresponding risk curves plotted in Figure 18. The total number of scenarios in all instances was 400.

Looking at Figure 18 one may notice that some risk curves are similar to those presented in the previous section; however, in this case all the curves lay below the curve with maximum *ENPV* (obtained with model PP) for small *NPV* values. Recall that when model RO-PP-FR was used to minimize financial risk at high *NPV* targets it produced solutions with high risk at small *NPV* values. These solutions were not found using model RO-PP-DR because they lead to such a small reduction in risk for large *NPV* values that the area under the risk curve (that is, the downside risk) turns out to be almost always higher than that of other solutions. Not finding these curves is not disadvantageous for this problem because they show no improvement from a risk perspective.

Two different kinds of probability distributions can be well identified in Figure 18: solutions for $\Omega = 600$–900 M$ behave as normally distributed random variables, whereas the rest of the solutions respond to a different kind of distribution. Three solutions have been selected to more clearly depict this observation in Figures 19 and 20. A schematic representation of the solutions for $\Omega = 900$ and 1100 is given in Figures 21 and 22, respectively.

Table 5. Product Stoichiometric Coefficients

Process	η_{ij} j_1	j_2	j_3	j_4	j_5	j_6	j_7	j_8
i_1	1							
i_2		1			1			
i_3			1					
i_4			1					
i_5				1			1	

Table 7. Expected Sale Prices and Demands

Market l_1 Chemical	Price (k$/kton) t_1	t_2	t_3	Demand (kton/yr) t_1	t_2	t_3
j_5	6000	6200	6400	75	90	105
j_6	14000	14500	15000	30	60	90
j_7	8000	8100	8200	80	85	90
j_8	24000	24200	24400	120	130	140

Table 8. Standard Deviation of Purchase Prices and Availabilities

Market l_1 Chemical	Price (k$/kton)			Availability (kton/yr)		
	t_1	t_2	t_3	t_1	t_2	t_3
j_1	1000	1075	1150	12.5	15	17.5
j_2	1250	1375	1500	15	20	25
j_3	300	305	310	2	2.1	2.2

A final observation based on Figure 17 is that as the *NPV* targeted is increased, the resulting risk curves approach to the curve with maximum *ENPV*. This is predicted by Eq. 25 of the first part, which establishes the relationship between downside risk and expected profit and shows that minimizing downside risk at high profit targets is equivalent to maximizing the expected profit.

Finally, one can conclude that using model RO-PP-DR allows managing financial risk because a set of solutions showing different risk curves was obtained. In the next section, this model is used to examine the effect of inventory on the risk curves.

Downside expected profit

The *DEP* is shown in Figure 23 for the optimal solution of model PP and a solution obtained using model RO-PP-FR with $\Omega = 900$, which has an expected value of M$ 908. Consider for instance any level of risk, say 50%. The risk downside expected profit *DEP*(x, 50%) is the expected profit with a level risk of 50%. Notice that in this way, the actual expected profit (*ENPV*) is given by *DEP*(x, 100%). For this reason, the PP solution (which has the maximum expected net present value) has the highest value of *DEP*(x, 100%). However, at other levels of confidence (from 0% up to about 67%) the solution for $\Omega = 900$ has a higher expected profit. This kind of plots, therefore, provides the decision maker additional insight about the risk exposure of each solution.

Computational Issues for Large-Scale Problems Using Model RO-SP-DR

Two methods are presented in this section aiming at obtaining improved computational performance when solving large-scale risk management problems with a large number of scenarios. First the Sampling Average Approximation method is introduced and later a solution algorithm that exploits the decomposable structure of model RO-SP-DR is described.

Sampling average approximation (SAA) method

In large-scale optimization of stochastic problems, sampling methods coupled with mathematical decomposition have been widely applied to many applications and incorporated in several algorithms [see Birge and Louveaux (1997), Higle and Sen (1996), and Infanger (1994) for reviews of these techniques]. Recently, Verweij et al. (2001) reported excellent computational results for different classes of large-scale stochastic routing problems using the sampling average approximation method (SAA). Additionally, the stochastic decomposition method (L-shaped) using Monte Carlo sampling (Higle and Sen, 1996) or importance sampling (Infanger, 1991) may show improved computational efficiency in some cases. In this article, the SAA method using Monte Carlo sampling was implemented and tested.

Table 9. Standard Deviation of Sale Prices and Demands

Market l_1 Chemical	Price (k$/kton)			Demand (kton/yr)		
	t_1	t_2	t_3	t_1	t_2	t_3
j_5	1500	1550	1600	18.75	22.5	26.25
j_6	3500	3625	3750	7.5	15	22.5
j_7	400	405	410	4	4.25	4.5
j_8	1200	1210	1220	6	6.5	7

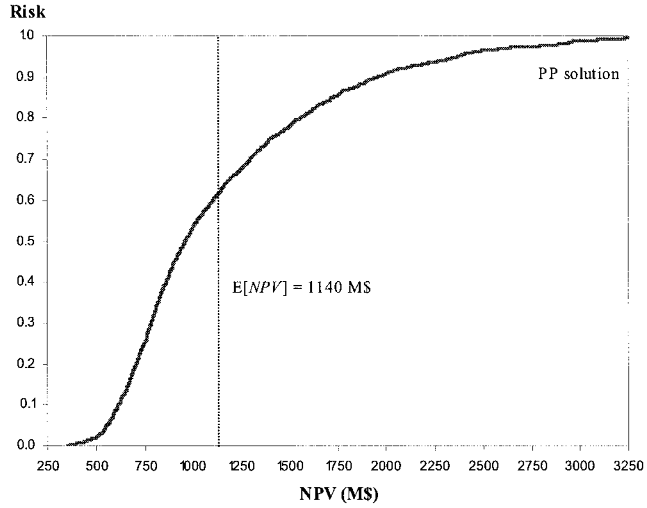

Figure 15. Solution that maximizes the expected net present value.

In the SAA technique, the expected second-stage profit (recourse function) in the objective function is approximated by an average estimate of *NS* independent random samples of the uncertain parameters, and the resulting problem is called the "approximation problem." Here, each sample corresponds to a possible scenario and so *NS* is the total number of scenarios considered. Then, the resulting approximation problem is solved repeatedly for *M* different independent samples (each of size *NS*) as a deterministic optimization problem. In this way, the average of the objective function of the approximation problems provides an estimate of the stochastic problem objective. Notice that this procedure may generate up to *M* different candidate solutions. To determine which of these *M* (or possibly less) candidates is optimal in the original problem, the values of the first-stage variables corresponding to each

Table 10. Solution Using Model PP

Process	Expanded at Period
i_1	t_1, t_3
i_2	t_3
i_3	t_1
i_4	t_2
i_5	t_2
E[*NPV*]	1140 M$
E[*Sales*]	5833 M$
E[*Purchases*]	3131 M$
E[*Operation Cost*]	1162 M$
Investment	400 M$

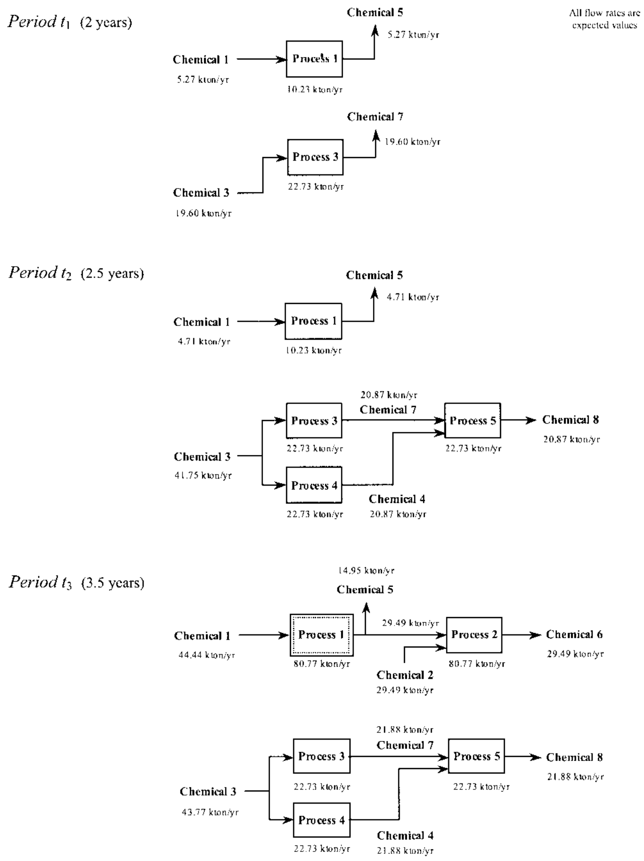

Figure 16. Solution that maximizes the expected net present value.

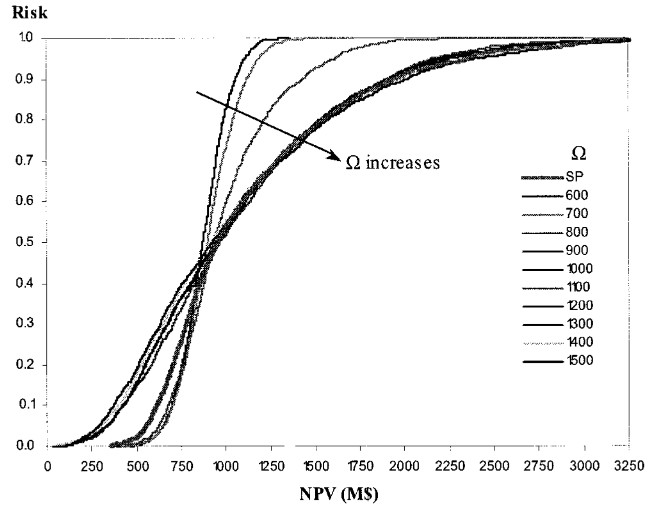

Figure 17. Solutions obtained using model RO-PP-FR.

candidate solution are fixed and the problem is solved again using a larger number of scenarios $NS' \gg NS$ to distinguish the candidates better. After solving these new problems, an estimate of the optimal solution of the original problem ($\hat{x}^*$) is obtained. Therefore, $\hat{x}^*$ is given by the solution of the approximate problems that yields the highest objective value for the approximation problem with NS' samples. This algorithm is presented next.

SAA algorithm

Select NS, NS', M
For $m = 1$ to M
For $s = 1$ to NS
Use Monte Carlo sampling to generate an independent observation of the uncertain parameters, $\omega_{s,m} = (q_s, T_s, h_s)$. Define $p_s = 1/NS$.
Next s
Solve problem RO-SP-DR with NS scenarios. Let the $\hat{x}^m$ be the optimal first-stage solution.
Next m
For $m = 1$ to M
For $s = 1$ to NS'
Use Monte Carlo sampling to generate an independent observation of the uncertain parameters, $\omega_{s,m} = (q_s, T_s, h_s)$. Define $p_s = 1/NS'$.
Next s
Solve problem RO-SP-DR with NS' scenarios, fixing $\hat{x}^m$ as the optimal first-stage solution.
Next m
Use $\hat{x}^* = \text{argmax}\{Obj(\hat{x}^m) \mid m = 1, 2, \ldots, M\}$ as the estimate of the optimal solution to the original problem where $Obj(\hat{x}^*)$ is the estimate of the optimal objective value.
End

Results

The performance of the above SAA algorithm was evaluated for the illustrative example considering the stochastic program-

Table 11. Solutions Obtained Using Formulation RO-PP-FR

Ω_1		PP	600	700	800	900	1000	1100	1200	1300	1400	1500
ρ_1		—	10000	10000	10000	10000	10000	10000	10000	10000	10000	10000
Process		*Period(s) in Which Capacity Expansion Is Performed*										
i_1		t_1, t_3	t_1	t_1	t_1, t_3	t_1, t_2	t_1, t_2	t_1, t_2	t_1, t_2	t_1, t_2	t_1, t_2	t_1
i_2		—	t_3	t_3	t_1, t_2	t_1, t_2	t_1, t_2	t_1, t_2	t_1, t_2	t_1, t_2	t_1, t_3	t_2
i_3		t_1	t_1	t_1	t_1	t_2	t_2	T_2	t_2	t_2	t_2	t_2
i_4		t_2	t_2	t_2	t_2	t_3	t_3	T_3	t_3	t_3	t_3	t_3
i_5		t_2	t_2	t_2	t_2	t_3	t_3	T_3	t_3	t_3	t_3	t_3
E[*NPV*]	M$	1140	876	906	1000	1064	1081	1081	1071	1066	1069	1075
E[*Sales*]	M$	5833	3693	3971	4702	6191	6220	6169	6201	6173	6314	6226
E[*Purchases*]	M$	3131	1947	2104	2509	3387	3387	3349	3381	3363	33495	3403
E[*Operation Cost*]	M$	1162	624	689	868	1343	1358	1345	1352	1348	1370	1362
Investment	M$	400	247	272	325	397	394	394	397	396	439	385

Table 12. Solutions Obtained Using Formulation RO-PP-DR (Downside Risk Approach)

	Ω											
Profit Target	PP	500	600	700	800	900	1000	1100	1200	1300	1400	1500
Process	*Period(s) in Which Capacity Expansion Is Performed*											
i_1	t_1, t_3	—	t_1	t_1	t_1	t_1	t_1, t_3	t_1, t_3	t_1, t_3	t_1, t_3	t_1, t_3	t_1, t_3
i_2	t_3	—	—	t_3	t_3	t_3	t_3	t_3	t_3	t_3	t_3	t_3
i_3	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1	t_1
i_4	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2
i_5	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2	t_2
E[*NPV*]	1140	855	875	897	908	9008	1032	1074	1099	1107	1122	1125
E[*Sales*]	5833	3472	3698	3916	3977	3981	4936	5221	5407	5378	5591	5610
E[*Purchases*]	3131	1843	1946	2072	2108	2110	2637	2795	287	2867	2999	3010
E[*Operation Cost*]	1162	551	631	675	689	690	932	1005	1052	1039	1099	1104
Investment	400	222	246	271	273	273	334	348	358	363	370	372

ming and the downside risk models. Eight different sample sizes (i.e., total number of scenarios) from 50 to 400 were considered and the problems for each sample were repeatedly solved 20 times ($M = 20$). Finally, the optimal first-stage solution (referred to here as Sol_1) was determined by solving the models for each of the candidate solutions with $NS' = 2000$. The rest of the candidate solutions are labeled Sol_2, Sol_3, and so forth, according to their frequency. For this problem, the structure of the candidate solutions is given by the binary variables Y_{it}, representing the decision of constructing or expanding process i at period t. For each model, the sample average and standard deviation of the *ENPV* were computed as follows

$$ENPV\ Average = \sum_{m=1}^{M} ENPV_m \tag{57}$$

$$ENPV\ Std\ Dev = \sqrt{\frac{\sum_{m=1}^{M} (ENPV_m - ENPV\ Average)^2}{M-1}} \tag{58}$$

Similar equations were used to compute the average and standard deviation for *DRisk*, as shown next

$$DRisk\ Average = \sum_{m=1}^{M} DRisk_m \tag{59}$$

$$DRisk\ Std\ Dev = \sqrt{\frac{\sum_{m=1}^{M} (DRisk_m - DRisk\ Average)^2}{M-1}} \tag{60}$$

The results for the different models are presented in Tables 13 through 15.

By analyzing the results presented in the above tables one can see that in this example the SAA method performs very well in terms of differentiating the optimal first-stage solution (Sol_1) from the rest of the candidates. Notice that in all cases the optimal solution appears with a much higher frequency than that of the rest of the candidates and that for samples with $NS \geq 150$ it is almost the only solution found. Additionally, in most cases only two solutions are selected as candidates, which is a

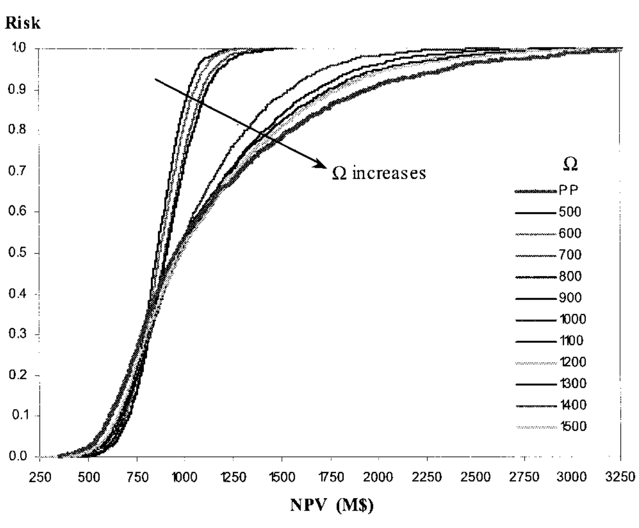

Figure 18. Solutions obtained using model RO-PP-DR.

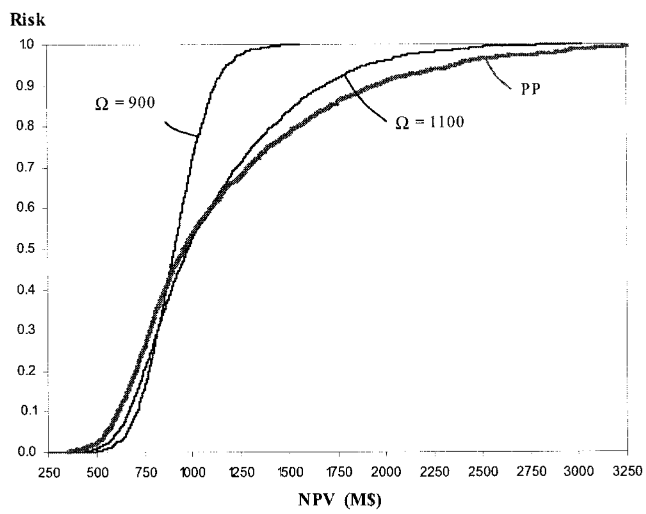

Figure 19. Selected solutions obtained using model RO-PP-DR.

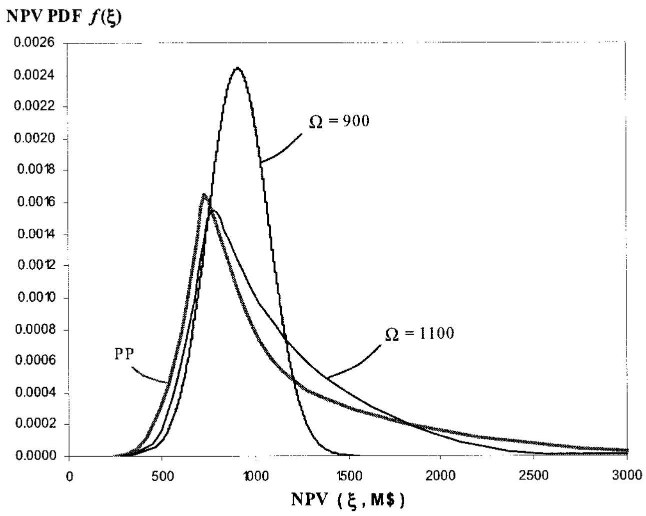

Figure 20. Probability distribution functions of selected solutions.

remarkably small number, considering that the possible arrangements could be as many as 2^{18}.

Another evident observation is that the standard deviation for both *ENPV* and *DRisk* decreases as the sample size increases; however, the rate of convergence does not seem very steep. In relative terms, the downside risk shows a higher

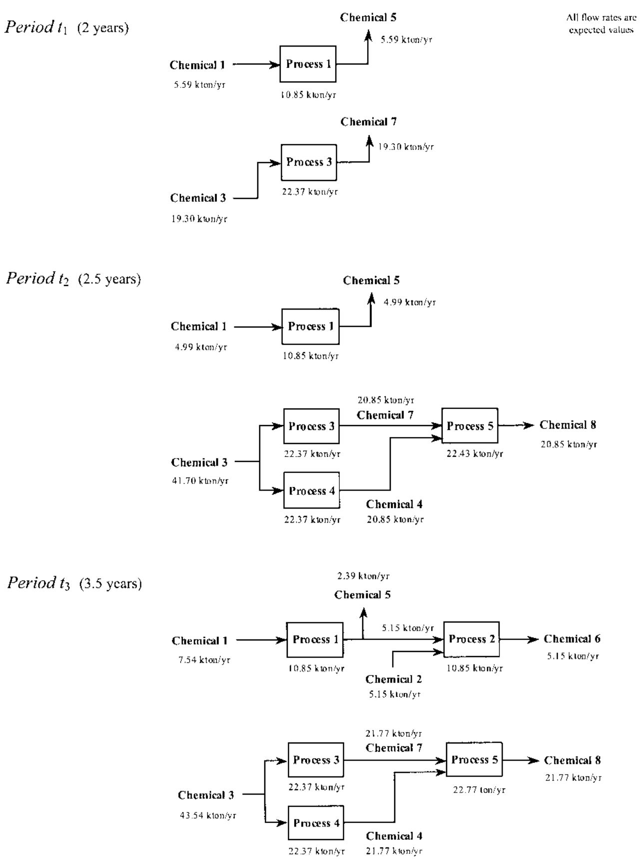

Figure 21. Solution obtained with model RO-PP-DR and $\Omega = 900$.

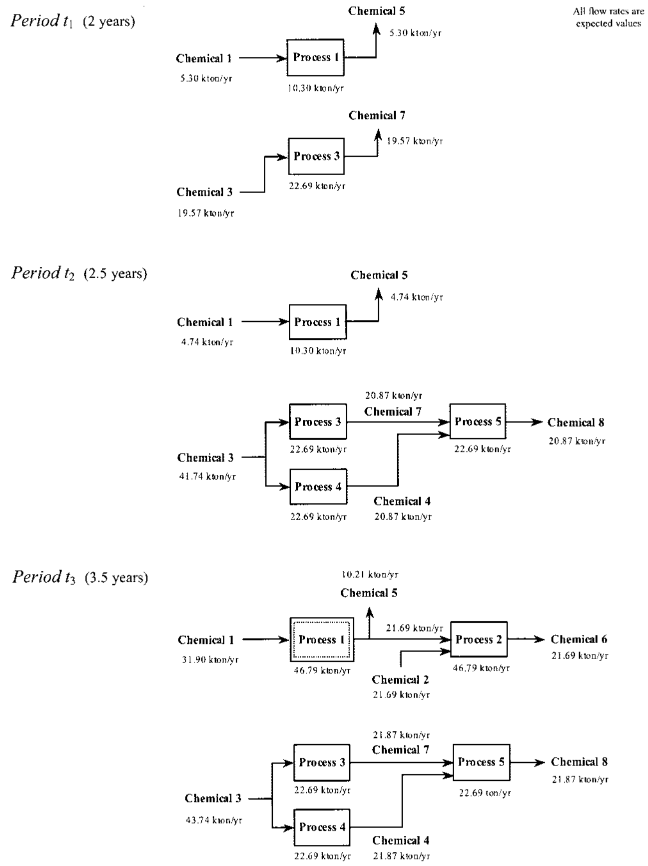

Figure 22. Solution obtained with model RO-PP-DR and $\Omega = 1100$.

standard deviation. Notice also that for samples with $NS \geq 300$ the expected net present value and the downside risk show little deviation from the optimal values obtained with $NS' = 2000$.

Generalized Benders Decomposition algorithm

For problems with a large number of scenarios and first-stage integer variables it is often thought that exploiting the

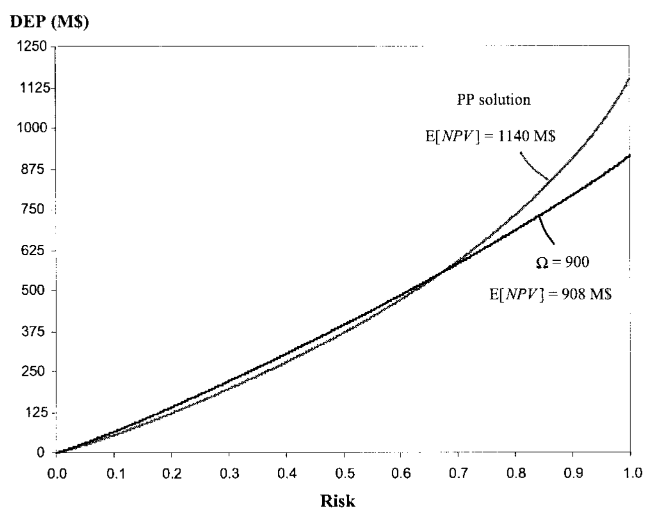

Figure 23. Downside expected profit of two different solutions.

Table 13. SAA Method Results for Model PP

NS NS′	M	ENPV Average	ENPV Std Dev	Candidate Solutions Frequency Sol_1	Sol_2	Sol_3	Sol_4
50	20	1177.1	78.3	13	4	2	1
100	20	1144.4	56.7	18	2	—	—
150	20	1144.7	31.9	20	—	—	—
200	20	1152.3	33.1	20	—	—	—
250	20	1156.2	30.5	20	—	—	—
300	20	1150.3	24.2	20	—	—	—
350	20	1157.6	30.8	20	—	—	—
400	20	1154.3	30.9	20	—	—	—
2000	—	1152.4	—	—	—	—	—

decomposable structure of the problem could help reduce the computational effort to solve it. In this sense, Generalized Benders Decomposition (GBD) (Benders, 1962; Geoffrion, 1972) has been found to be efficient to solve some MILP problems arising in stochastic programming (Liu and Sahinidis, 1996). Generally, the computational efficiency of this algorithm is highly dependent on the specific structure and characteristics of the problem. Hence, strategies that combine the GBD algorithm with other approaches such as branch and cut (Verweij et al., 2001), addition of integer cuts (Iyer and Grossman, 1989), or heuristics (Ahmed and Sahinidis, 2000a), designed for the specific application, may greatly improve computational performance.

As a first step, the incorporation of GBD to solve large-scale instances of risk management model RO-SP-DR in the proposed SAA algorithm is suggested, and two different GBD algorithms are explored. In the first one, the first-stage variables are taken as complicating variables, rendering a Benders' Primal problem with only second-stage constraints and variables, which is then solved simultaneously for all scenarios. This algorithm will be referred to in this article as "*Benders*." In addition, the second GBD algorithm also considers the first-stage variables as complicating ones, although the resulting Benders' Primal problem is solved separately for each scenario, taking advantage of the decomposable structure of the problem. This algorithm is referred to here as "*Benders-SD*" and corresponds to a stochastic decomposition of the original problem. Before presenting the structure of the GBD algorithms, the optimization problems that are used inside the algorithm are defined.

Optimality Primal P(k)

$$\text{Max } \mu \sum_{s\in S} p_s q_s^T y_s^k - \sum_{s\in S} p_s \delta_s^k \tag{61}$$

s.t.

$$W y_s^k = h_s - T_s x^k \qquad \forall \; s \in S \tag{62}$$

$$\delta_s^k + q_s^T y_s^k \geq \Omega + c^T x^k \qquad \forall \; s \in S \tag{63}$$

$$y_s^k,\; \delta_s^k \geq 0 \qquad \forall \; s \in S \tag{64}$$

Feasibility Primal F(k)

$$\text{Min} \sum_{s\in S} [e^T \upsilon_s^+ + e^T \nu_s + e^T \upsilon_s^-] \tag{65}$$

s.t.

$$W y_s^k + I\upsilon_s^+ - I\upsilon_s^- = h_s - T_s x^k \qquad \forall \; s \in S \tag{66}$$

$$\delta_s^k + q_s^T y_s^k + I\nu_s \geq \Omega + c^T x^k \qquad \forall \; s \in S \tag{67}$$

$$y_s^k,\; \delta_{si}^k,\; \upsilon_s^+,\; \upsilon_s^-,\; \nu_s \geq 0 \qquad \forall \; s \in S \tag{68}$$

Master Problem M(k)

$$\text{Max } \theta \tag{69}$$

s.t.

$$A x^k = b \tag{70}$$

Table 14. SAA Method Results for Model RO-PP-DR (Ω = 900, μ = 0.001)

NS NS′	M	ENPV Average	ENPV Std Dev	DRisk Average	DRisk Std Dev	Candidate Solutions Frequency Sol_1	Sol_2	Sol_3	Sol_4
50	20	972.7	78.5	53.1	12.7	8	6	4	2
100	20	932.9	42.5	57.0	7.1	15	5	—	—
150	20	917.6	28.7	56.5	5.3	19	1	—	—
200	20	927.4	48.7	58.6	4.0	16	3	—	1
250	20	931.0	36.6	57.6	6.2	16	4	—	—
300	20	919.6	27.8	57.6	5.1	18	2	—	—
350	20	924.5	33.2	56.6	5.3	18	2	—	—
400	20	915.3	22.5	58.6	4.1	18	2	—	—
2000	—	914.2	—	58.1	—	1	—	—	—

Table 15. SAA Method Results for Model RO-PP-DR ($\Omega = 1100$, $\mu = 0.001$)

NS *NS'*	*M*	*ENPV Average*	*ENPV Std Dev*	*DRisk Average*	*DRisk Std Dev*	Candidate Solutions Frequency		
						Sol_1	Sol_2	Sol_3
50	20	1055.5	74.5	176.8	25.9	14	4	2
100	20	1055.5	59.2	180.0	23.6	19	1	—
150	20	1076.4	36.5	172.0	13.5	20	—	—
200	20	1073.7	47.2	175.0	17.8	20	—	—
250	20	1074.8	35.0	171.8	14.2	20	—	—
300	20	1088.2	23.3	167.8	11.6	20	—	—
350	20	1086.1	27.9	169.0	10.5	20	—	—
400	20	1081.0	28.4	172.0	9.3	20	—	—
2000	—	1092.9	—	168.2	—	1	—	—

$$\theta \leq \mu c^T x^k + \sum_{s \in S} [\pi_s^{1,k}(h_s - T_s x^k) + \pi_s^{2,k}(\Omega + c^T x^k)] \qquad k \in K_o \quad (71)$$

$$0 \leq \sum_{s \in S} [\sigma_s^{1,k}(h_s - T_s x^k) + \sigma_s^{2,k}(\Omega + c^T x^k)] \qquad k \in K_f \quad (72)$$

$$x^k \geq 0 \quad x^k \in X \quad (73)$$

The above problems are used in each iteration (k) of the *Benders* algorithm. The first problem, P(k) or *Optimality Primal* is used to generate an optimality cut constraint that is a linear approximation of the recourse function. If subproblem P(k) turns out to be infeasible, then a feasibility primal problem F(k) is solved to generate a feasibility cut that reduces the feasible region of the master problem, eliminating infeasibilities in the second-stage. Finally, the master problem M(k) is iteratively solved to update the first-stage decisions until optimality is achieved.

The optimization problems for the *Benders-SD* algorithm are presented next. In this case, the optimality and feasibility problems are solved independently for each different scenario.

Optimality Primal P(k, s)

$$\text{Max } \mu q_s^T y_s^k - \delta_s^k \quad (74)$$

s.t.

$$W y_s^k = h_s - T_s x^k \quad (75)$$

$$\delta_s^k + q_s^T y_s^k \geq \Omega + c^T x^k \quad (76)$$

$$y_s^k,\ \delta_s^k \geq 0 \quad (77)$$

Feasibility Primal F(k, s)

$$\text{Min } e^T \upsilon_s^+ + e^T \nu_s + e^T \upsilon_s^- \quad (78)$$

s.t.

$$W y_s^k + I\upsilon_s^+ - I\upsilon_s^- = h_s - T_s x^k \quad (79)$$

$$\delta_s^k + q_s^T y_s^k + I\nu_s \geq \Omega + c^T x^k \quad (80)$$

$$y_s^k,\ \delta_{si}^k,\ \upsilon_s^+,\ \upsilon_s^-,\ \nu_s \geq 0 \quad (81)$$

Master Problem M(k)

$$\text{Max } \theta \quad (82)$$

s.t.

$$A x^k = b \quad (83)$$

$$\theta \leq \mu c^T x^k + \sum_s p_s[\pi_s^{1,k}(h_s - T_s x^k) + \pi_s^{2,k}(\Omega + c^T x^k)] \qquad k \in K_o \quad (84)$$

$$0 \leq \sigma_s^{1,k}(h_s - T_s x^k) + \sigma_s^{2,k}(\Omega + c^T x^k) \qquad s \in S_k,\ k \in K_f \quad (85)$$

$$x^k \geq 0 \quad x^k \in X \quad (86)$$

Having properly defined the correspondent optimization problems, the GBD algorithms are presented next.

Benders algorithm

Initialization Set $k \to 1$ and choose $x^1 \in \{x \mid Ax = b, x \geq 0\}$. Set $UB \to \infty$ and $LB \to -\infty$. Let the set of optimality and feasibility cuts be $K_o \to \{\emptyset\}$ and $K_f \to \{\emptyset\}$, respectively.

Step 1 Solve optimality primal P(k).

If P(k) is feasible, then:

- Let $\pi_s^{1,k}$ and $\pi_s^{2,k}$ be the optimal Lagrange multipliers associated with constraints 62 and 63, respectively.
- Set $K_o \to K_o \cup \{k\}$.
- Let y_s^{k*} and δ_s^{k*} be the optimal values of the variables in P(k) and go to Step 2.

Else,

- Solve the feasibility cut primal F(k).
- Let $\sigma_s^{1,k}$ and $\sigma_s^{2,k}$ be the optimal Lagrange multipliers associated with constraints 66 and 67, respectively.
- Set $K_f \to K_f \cup \{k\}$ and go to Step 3.

Step 2 Let $LB = \max\{LB,\ \mu(\sum_s p_s q_s^T y_s^{k*} - c^T x^k) - \sum_s p_s \delta_s^{k*}\}$. If $UB - LB <$ tolerance, then terminate with x^k being an optimal solution to the problem.

Table 16. Computational Performance for Model PP

	CPLEX			BENDERS			BENDERS-SD		
NS	Total Time (s)	Optimization	Processing	Total Time (s)	Optimization	Processing	Total Time (s)	Optimization	Processing
50	14	13	1	175	120	55	1798	932	866
100	77	76	1	272	181	91	3835	1800	2035
150	169	167	2	297	189	108	5768	2597	3171
200	344	341	3	427	268	159	—	—	—
250	466	462	4	427	256	171	—	—	—
300	631	626	5	460	269	191	—	—	—
350	821	814	7	554	326	228	—	—	—
400	1327	1319	8	762	453	310	—	—	—

Step 3 Solve the master problem M(k). If this problem is infeasible, then terminate; the original problem is infeasible. Else, let x^{k+1} be a new first-stage decision and $UB = \theta^*$ be a new upper bound of the solution. Set $k \rightarrow k + 1$ and go to Step 1.

Benders-SD algorithm

Initialization Set $k \rightarrow 1$ and choose $x^1 \in \{x \mid Ax = b, x \geq 0\}$. Set $UB \rightarrow \infty$ and $LB \rightarrow -\infty$. Let the set of optimality and feasibility cuts be $K_o \rightarrow \{\emptyset\}$ and $K_f \rightarrow \{\emptyset\}$, respectively.

Step 1 For each s solve optimality primal P(k, s).

If P(k, s) is feasible $\forall$ s, then:

- Let $\pi_s^{1,k}$ and $\pi_s^{2,k}$ be the optimal Lagrange multipliers associated with constraints 75 and 76, respectively.
- Set $K_o \rightarrow K_o \cup \{k\}$.
- Let y_s^{k*} and δ_s^{k*} be the optimal values of the variables in P(k, s) and go to Step 2.

Else,

- Let S_k be the set scenarios for which P(k, s) is infeasible.
- Solve the feasibility cut primal F(k, s) $\forall$ $s \in S_k$.
- Let $\sigma_s^{1,k}$ and $\sigma_s^{2,k}$ be the optimal Lagrange multipliers associated with constraints 79 and 80, respectively.
- Set $K_f \rightarrow K_f \cup \{k\}$ and go to Step 3.

Step 2 Let $LB = \max\{LB, \mu(\Sigma_s\, p_s q_s^T y_s^{k*} - c^T x^k) - \Sigma_s\, p_s \delta_s^{k*}\}$. If $UB - LB <$ tolerance, then terminate with x^k being an optimal solution to the problem.

Step 3 Solve the master problem M(k). If this problem is infeasible, then terminate; the original problem is infeasible. Else, let x^{k+1} be a new first-stage decision and $UB = \theta^*$ be a new upper bound of the solution. Set $k \rightarrow k + 1$ and go to Step 1.

Results

The performance of the GBD algorithms was evaluated for the example using the stochastic programming (PP) and the downside risk (RO-PP-DR) models, considering eight different numbers of scenarios ranging from 50 to 400. The algorithms were coded in GAMS (Brooke et al., 1988) and benchmarked with CPLEX 7.0 (CPLEX, 2000) using default options. In each case, the total computational time to solve the problem was accounted for. In addition, the time required to solve the optimization problems and time consumed in pre- and postprocessing tasks was also recorded. This latter is the accumulation of times of startup, compilation, execution, and closedown for each subproblem. The results are summarized in Tables 16 to 18 and Figures 24 to 26.

Some insightful observations can be derived from the results presented in the previous tables and figures. First, we can conclude that Benders-SD is by far the most inefficient technique for this test problem using GAMS. This is mainly a consequence of the excessive time required for pre- and postprocessing tasks such as startup, compilation, execution, and closedown for each subproblem. Notice that this time is in most cases larger than the actual time required to solve the problems to optimality. Apparently, GAMS is not very efficient for these tasks and the pre- and postprocessing times could be reduced if another platform were used.

Another observation is that the Benders algorithm becomes more efficient as the number of scenarios increases. Even though the total time is always increasing with the number of scenarios, the slope for this technique seems to be rather flat. This becomes more evident by looking at Figures 24 to 26 and noticing that the curves for Benders do not only lie below those for CPLEX but also have a flatter increase trend.

Finally, it should be noted that CPLEX solver is very efficient for problems with few scenarios ($NS < 200$). This is a very important property in view of the results obtained by use of the SAA algorithm presented in the previous section. Because the SAA algorithm showed very good performance in terms of differentiating the optimal first-stage solution, even for problems with small number of scenarios, one promising strategy to reduce the computational times would be to consider a large number of repetitions, using CPLEX to solve

Table 17. Computational Performance for Model RO-PP-DR ($\Omega = 900$, $\mu = 0.001$)

	CPLEX			BENDERS			BENDERS-SD		
NS	Total Time (s)	Optimization	Processing	Total Time (s)	Optimization	Processing	Total Time (s)	Optimization	Processing
50	33	32	1	274	198	76	2271	1160	1111
100	152	151	1	369	254	115	4600	2191	2409
150	302	300	2	385	254	131	5886	2570	3316
200	644	641	3	443	281	162	—	—	—
250	1001	997	4	575	362	213	—	—	—
300	2081	2076	5	612	375	237	—	—	—
350	1459	1449	10	836	523	313	—	—	—
400	3550	3541	9	866	526	340	—	—	—

Table 18. Computational Performance for Model RO-PP-DR ($\Omega = 1100$, $\mu = 0.001$)

	CPLEX			BENDERS			BENDERS-SD		
NS	Total Time (s)	Optimization	Processing	Total Time (s)	Optimization	Processing	Total Time (s)	Optimization	Processing
50	18	17	1	254	186	68	2015	1015	1000
100	94	93	1	353	246	107	4248	2009	2239
150	116	114	2	433	287	146	7013	3075	3938
200	273	270	3	489	319	170	—	—	—
250	477	473	4	623	397	226	—	—	—
300	569	564	5	614	380	234	—	—	—
350	731	725	6	843	526	317	—	—	—
400	985	977	8	762	458	304	—	—	—

problems with fewer scenarios. The computational assessment of such a strategy is left for future work.

Conclusions

This article has addressed several issues related to financial risk management in the framework of two-stage stochastic programming models, developing new formulations that allow the decision maker to obtain solutions that are in agreement with his/her risk preference. First, the theoretical aspects of risk management, including a formal definition of risk as well as mathematical models that manage financial risk, were developed. The trade-offs between risk and profitability were discussed and the cumulative risk curves were found to be an appropriate way to visualize the risk behavior of different alternatives. Furthermore, the concept of downside risk was examined, finding a close relationship with financial risk. Consequently, it was suggested that downside risk be used to measure financial risk, given that it eliminates the need to introduce new binary variables that increase the computational burden. An illustrative example showed that the maximization of the expected net present value by itself is not an appropriate objective and that solutions with higher risk exposure are obtained. Also, a preliminary study of the computation issues related to risk management for large-scale problems was presented. In this regard, the sample average approximation method was tested obtaining promising results. Additionally, two Generalized Benders Decomposition algorithms were presented, fundamentally because they represent the theoretical foundations for future research of more advanced stochastic decomposition methods. For the test problem, it was observed that the use of GBD without scenario decomposition requires less computational time than the CPLEX standard optimization solver when the number of scenarios increases. On the other hand, GBD with scenario decomposition had a poor performance, mainly because the pre- and postprocessing times become increasingly large for problems with a large number of scenarios. Finally, it was observed that CPLEX solver is very efficient for problems with few scenarios ($NS < 200$), suggesting that using this solver in the SAA algorithm with large M and small NS could be computationally efficient.

Notation

Indices

i = for the set of processes (i = 1 to NP)
j = for the set of chemicals (j = 1 to NC)
k = Bender's Decomposition iteration index
l = for the set of markets (l = 1 to NM)
t = for the set of time periods (t = 1 to NT)
s = for the set of scenarios (s = 1 to NS)

Sets

I = profit targets

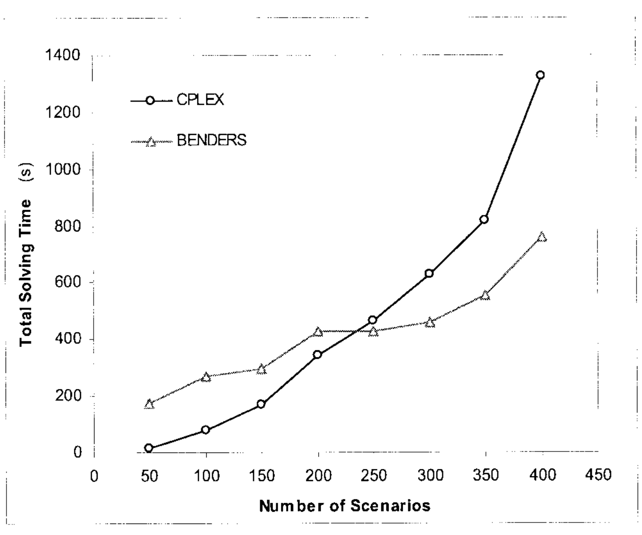

Figure 24. Computational performance for model PP.

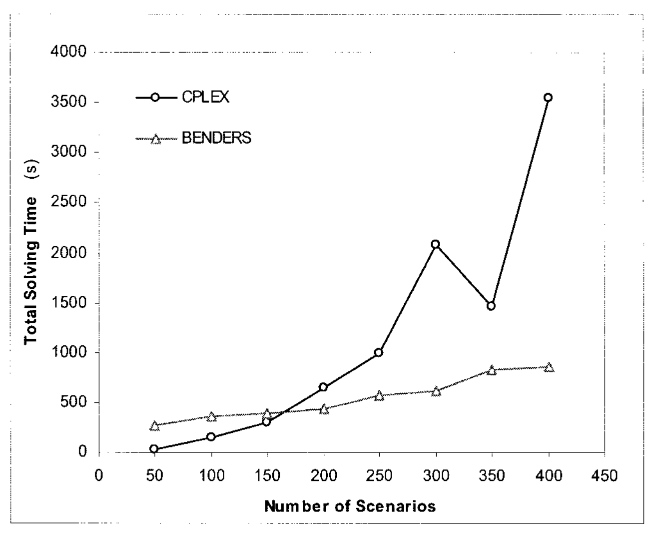

Figure 25. Computational performance for model RO-PP-DR ($\Omega = 900$, $\mu = 0.001$).

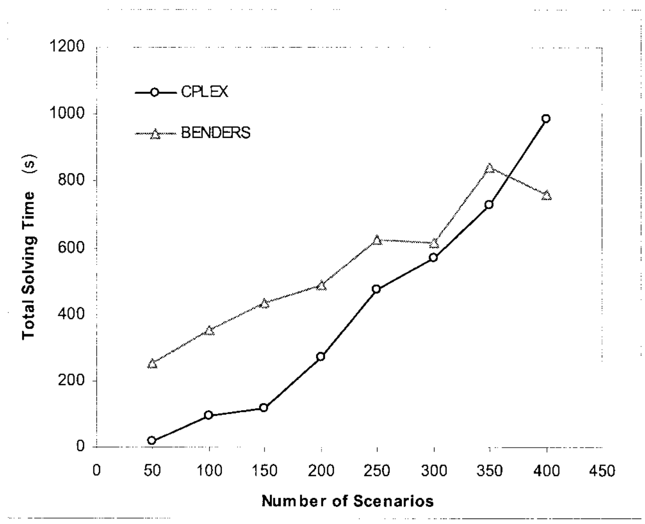

Figure 26. Computational performance for model RO-PP-DR (Ω = 1100, μ = 0.001).

K_o = set of optimality cuts in the Generalized Benders Decomposition algorithm
K_f = set of feasibility cuts in the Generalized Benders Decomposition algorithm
S = set of scenarios
S_k = set of scenarios with infeasible optimality sub problems at iteration k in GBD
X = feasibility set for first-stage decision variables

Parameters

A = matrix of deterministic coefficients of the first-stage constraints
a^L_{jlts} = lower bound for purchases of chemical j in market l within period t under scenario s
a^U_{jlts} = upper bound for purchases of chemical j in market l within period t under scenario s
b = vector of deterministic independent terms of the first-stage constraints
c = vector of deterministic first-stage cost coefficients
CI_t = maximum capital investment allowed in period t
d^L_{jlts} = lower bound for sales of chemical j in market l within period t and under scenario s
d^U_{jlts} = upper bound for sales of chemical j in market l within period t under and scenario s
E^L_{it} = lower bound for the expansion capacity of process i at the beginning of period t
E^U_{it} = upper bound for the expansion capacity of process i at the beginning of period t
h_s = vector of stochastic independent terms of the second-stage constraints
L_t = length of period t (in years)
LB = objective function lower bound for GBD algorithm
$NEXP_i$ = maximum number of expansions allowed for process i
p_s = probability of occurrence of scenario s
q_s = vector of stochastic coefficients of the recourse function
T_s = technology matrix of the second-stage constraints
UB = objective function upper bound for the GBD algorithm
α_{it} = expansion cost per unit of capacity for process i at the beginning of period t
β_{it} = fixed cost of establishing or expanding process i at the beginning of period t
γ_{jlts} = sales price of chemical j in market l within period t under scenario s
δ_{it} = operating cost coefficient of process i within period t under scenario s
Γ_{jlt} = purchase price of chemical j in market l within period t under scenario s
η_{ij} = stoichiometric coefficient representing the amount of chemical j produced per unit of capacity of process i
μ_{ij} = stoichiometric coefficient representing the amount of chemical j consumed per unit of capacity of process i
μ = goal programming weight for downside risk formulations
ρ = goal programming weight for financial risk formulations
ε = upper bound for the upper partial mean or financial risk in RR formulations
Ω = profit target
$\bar{\xi}$ = profit upper bound
$\underline{\xi}$ = profit lower bound

Variables

E_{it} = expansion in capacity of process i at the beginning of period t
$ENPV$ = expected net present value
P_{jlt} = units of chemical j purchased in market l within period t under scenario s
Q_{it} = capacity of process i at the beginning of period t
S_{jlt} = units of chemical j sold in market l within period t under scenario s
W_{it} = operating capacity of process i at the beginning of period t under scenario s
Y_{it} = binary variable set to one only if process i is expanded at the beginning of period t
x = first-stage decision variables. The values of these variables define a "design" or "plan"
y_s = second-stage decision variables for scenario s
z_{si} = binary variable equal 1 if the profit of scenario s is smaller than the profit target Ω_i
δ_s = positive profit deviation from the target for scenario s
$\pi^{1,k}_s$, $\pi^{2,k}_s$ = optimal Lagrange multipliers for optimality subproblem of GBD
υ^+_s, υ^-_s, ν_s = variables for feasibility subproblem of GBD
$\sigma^{1,k}_s$, $\sigma^{2,k}_s$ = optimal Lagrange multipliers for feasibility subproblem of GBD
θ = objective function for master problem of GBD
ξ = profit

Functions

$DRisk(x, \Omega)$ = downside risk of solution x at a profit target Ω
$f(x, \Omega)$ = profit probability distribution function
$Profit(x)$ = profit for design x
$Risk(x, \Omega)$ = financial risk of solution x at a profit target Ω
$DEP(x, p_\Omega)$ = downside Expected Profit for a risk level p_Ω
$VaR(x, p_\Omega)$ = value at Risk for a risk level p_Ω
$z_s(x, \Omega)$ = probability of the profit for design x under scenario s being lower than Ω
$\delta(x, \Omega)$ = positive deviation from the profit target Ω for design x

Literature Cited

Ahmed, S., and N. V. Sahinidis, "Robust Process Planning Under Uncertainty," *Ind. Eng. Chem. Res.*, **37**(5), 1883 (1998).
Ahmed, S., and N. V. Sahinidis, "Analytical Investigations of the Process Planning Problem," *Comput. Chem. Eng.*, **23**(11–12), 1605 (2000a).
Ahmed, S., and N. V. Sahinidis, "Selection, Acquisition and Allocation of Manufacturing Technology in a Multi-Product Environment, *Manage. Sci.*, in press (2000b).
Ahmed, S., Tawarmalani M., and N. V. Sahinidis, "A Finite Branch and Bound Algorithm for Two-Stage Stochastic Integer Programming," *Math. Program.*, in press (2000).
Applequist, G., J. F. Pekny, and G. V. Reklaitis, "Risk and Uncertainty in Managing Chemical Manufacturing Supply Chains," *Comput. Chem. Eng.*, **24**(9/10), 2211 (2000).
Barbaro, A., and M. Bagajewicz, *Financial Risk Management in Planning under Uncertainty. Proc of Fourth Int Conf on Foundations of Computer-Aided Process Operations (FOCAPO 2003),* Ignacio E. Grossmann, and

Conor M. McDonald, eds, ISBN-0965589110, Coral Springs, FL (Jan., 2003).
Beale, E. M. L., "On Minimizing a Convex Function Subject to Linear Inequalities," *J. Royal Stat. Soc.*, Series B **17**, 173 (1955).
Bellman, R., and L. A. Zadeh, "Decision-making in a Fuzzy Environment," *Manage. Sci.*, **17**, 141 (1970).
Benders, J. F., "Partitioning Procedures for Solving Mixed-Variables Programming Problems," *Numerische Mathematik,* **4**, 238 (1962).
Berman, O., and Z. Ganz, "The Capacity Expansion Problem in the Service Industry," *Comput. Operations Res.*, **21**, 557 (1994).
Birge, J. R., and Louveaux F., *Introduction to Stochastic Programming*, Springer-Verlag, New York (1997).
Brooke, A., D. Kendrick, and A. Meeraus, *GAMS—A User's Guide*, The Scientific Press, Redwood City, CA (1988).
Caroe, C. C, and R. Schultz, "Dual Decomposition in Stochastic Integer Programming," *Operations Res. Lett.*, **24**, 37-45 (1999).
Charnes, A., and W. W. Cooper, "Chance-Constrained Programming," *Manage. Sci.*, **6**, 73 (1959).
Cheng, L., E. Subrahmanian, and A. W. Westerberg, "Design and Planning under Uncertainty: Issues on Problem Formulation and Solution," *Comput. Chem. Eng.*, **27**(6), 781 (2003).
Cheung, R. K. M., and W. B. Powel, "Shape—A Stochastic Hybrid Approximation Procedure for Two-Stage Stochastic Programs," *Operations Res.*, **48**(1), 73 (2000).
CPLEX. CPLEX 7.0 User's Manual. ILOG CPLEX Division. Incline Village, NV (2000).
Dantzig, G. B., "Linear Programming under Uncertainty," *Manage. Sci.*, **1**, 197 (1955).
Dupacova, J., and W. Romisch, "Quantitative Stability for Scenario-Based Stochastic Programs," *Prague Stochastics*, **98**, S. 119, JCMF, Praha (1998).
Eppen, G. D., Martin R. K., and L. Schrage, "A Scenario Approach to Capacity Planning," *Operations Res.*, **37**, 517 (1989).
Geoffrion, A. M., "Generalized Benders Decomposition," *J. Optimization Theory Appl.*, **10**, 237 (1972).
Guldimann, T., "The Story of Risk Metrics," *Risk*, **13**(1), 56 (2000).
Gupta, A., and C. Maranas. "Market-Based Pollution Abatement Strategies: Risk Management Using Emission Option Contracts," *Ind. Eng. Chem. Res.*, **42**, 802 (2003).
Higle, J. L., and S. Sen, *Stochastic Decomposition. A Statistical Method for Large Scale Stochastic Linear Programming*, Kluwer Academic Publishers, Norwell, MA (1996).
Ierapetritou, M. G., and E. N. Pistikopoulos, "Simultaneous Incorporation of Flexibility and Economic Risk in Operational Planning under Uncertainty," *Comput. Chem. Eng.*, **18**(3), 163 (1994).
Infanger, G., "Monte Carlo (Importance) Sampling within a Benders Decomposition Algorithm for Stochastic Linear Programs (Extended Version: Including Results of Large-Scale Problems)," Technical Report SOL 91-6, Systems Optimization Laboratory, Stanford University, CA (1991).
Infanger, G., *Planning under Uncertainty: Solving Large-Scale Stochastic Linear Programs*, Boyd and Fraser, Danvers, MA (1994).
Iyer, R. R., and I. E. Grossmann, "A Bilevel Decomposition Algorithm for Long-Range Planning of Process Networks," *Ind. Eng. Chem. Res.*, **37**, 474 (1998).
Jorion, P., *Value at Risk. The New Benchmark for Managing Financial Risk*, 2nd Edition, McGraw Hill, New York (2000).
Kall, P., and S. W. Wallace, *Stochastic Programming*, Wiley, Chichester, UK (1994).
Linsmeier, T. J., and N. D. Pearson, "Value at Risk," *Financial Analysts J.*, **56**(2), March/April (2000).
Liu, M. L., and N. V. Sahinidis, "Optimization in Process Planning under Uncertainty," *Ind. Eng. Chem. Res.*, **35**, 4154 (1996).
Marti, K., and P. Kall, eds., *Stochastic Programming Methods and Technical Applications, LNEMS*, Vol. 458, Springer-Verlag, Berlin-Heidelberg (1998).
McCray, A. W. *Petroleum Evaluations and Economic Decisions*, Prentice Hall, Englewood Cliffs, NJ (1975).
Mulvey, J. M., R. J. Vanderbei, and S. A. Zenios, "Robust Optimization of Large-Scale Systems," *Operations Res.*, **43**, 264 (1995).
Murphy, F. H., S. Sen, and A. L. Soyster, "Electric Utility Expansion Planning in the Presence of Existing Capacity: A Nondifferentiable, Convex Programming Approach," *Comput. Operations Res.*, **14**, 19 (1987).
Pistikopoulos, E. N., and M. G. Ierapetritou, "Novel Approach for Optimal Process Design under Uncertainty," *Comput. Chem. Eng.*, **19**(10), 1089 (1995).
Sahinidis, N. V., I. E. Grossmann, R. E. Fornari, and M. Chathrathi, "Optimization Model for Long Range Planning in the Chemical Industry," *Comput. Chem. Eng.*, **13**, 1049 (1989).
Takriti, S., and S. Ahmed, "On Robust Optimization of Two-Stage Systems," Mathematical Programming, On-line Publication. DOI 10.1007/s10107-003-0373-y (2001).
Uryasev, S., and P. M. Pardalos, eds., *Stochastic Optimization: Algorithm and Applications*, Kluwer Academic, Norwell, MA (2001).
Verweij, B., S. Ahmed, A. J. Kleywegt, G. Nemhauser, and A. Shapiro, "The Sample Average Approximation Method Applied to Stochastic Routing Problems: A Computational Study," *Comput. Appl. Optimization*, **24**, 289 (2003).
Zimmermann, H. J., *Fuzzy Sets and Decision Making, and Expert Systems*, Kluwer Academic Publishers, Boston (1987).

Appendix

A. Continuous behavior of the cumulative risk curves

This appendix proves formally that, if *Profit*(x) has a continuous probability distribution, then, when the number of scenarios becomes increasingly large [$Cardinality(S) \to \infty$], the cumulative risk curves calculated using scenarios approach continuous behavior.

Mathematically, the mentioned observation comes from the fact that the continuous and discrete definitions of financial risk yield the same value, that is

$$\lim_{Card(S)\to\infty} \sum_s p_s z_s(x, \Omega) = \int_{-\infty}^{\Omega} f(x, \xi) d\xi \tag{A1}$$

Using the definition of risk, one can relate the probability distribution function $f(x, \xi)$ with a probability as follows

$$Risk(x, \Omega) = P(\xi < \Omega) = \int_{-\infty}^{\Omega} f(x, \xi) d\xi \tag{A2}$$

Now assume that one takes NS random samples of the uncertainty space using the probability distributions of the uncertain parameters for the problem. Then, each sample s constitutes a possible *scenario* with a probability of occurrence defined as $p_s = 1/NS$, rendering $\sum_{s \in S} \mathrm{p_s} = 1$.

From probability theory, it is well known that when the number of samples (scenarios) becomes increasingly large, the sample distribution approximates the actual distribution of the original probability space. Then, the profit distribution obtained using the sampled parameters will approximate to the actual profit distribution.

Assume now that one has solved the two-stage stochastic problem with NS scenarios obtaining the resulting profit for each scenario, ξ_s, among which there are N different values, with $N \leq NS$. Afterward, the scenarios were sorted in ascending profit order such that $\xi_{n+1} > \xi_n$ for $n = 1$ to N. In turn, the total number of scenarios having a profit less or equal than the target Ω is denoted as N_Ω. Because of the convergence of the sample distribution to the actual distribution of the uncertainty space, it follows that when the profit has a continuous probability distribution and $NS \to \infty$, then $N \to \infty$, $N\Omega \to \infty$, and the difference $\Delta\xi_n = \xi_{n+1} - \xi_n$ tends to zero.

Now, given that $f(x, \xi)$ is a continuous function, the Mean Value Theorem may be used to approximate its value at any given value of profit ξ. Then, from Eq. A2, one obtains

$$f(x, \xi_n) = \lim_{NS\to\infty} \frac{P(\xi < \xi_{n+1}) - P(\xi < \xi_n)}{(\xi_{n+1} - \xi_n)} \tag{A3}$$

From probability theory, it follows that the numerator on the right-hand side of Eq. A3 is just the probability of the profit being between ξ_n and ξ_{n+1} [i.e., $P(\xi_n \le \xi < \xi_{n+1})$]. Using this result, the integral defining financial risk in Eq. A2 becomes

$$\begin{aligned}\int_{-\infty}^{\Omega} f(x, \xi)d\xi &= \lim_{NS\to\infty} \sum_{n=1}^{N\Omega} \left[\frac{P(\xi_n \le \xi < \xi_{n+1})}{\Delta\xi_n}\right]\Delta\xi_n \\ &= \lim_{NS\to\infty} \sum_{n=1}^{N\Omega} P(\xi_n \le \xi < \xi_{n+1})\end{aligned} \tag{A4}$$

In order to calculate the probability $P(\xi_n \le \xi < \xi_{n+1})$, we define the set α_n as follows: $\alpha_n = \{s \mid \xi_n \le \xi_s < \xi_n\}$. Then

$$P(\xi_n \le \xi < \xi_{n+1}) = \lim_{NS\to\infty} \frac{Card(\alpha_n)}{NS} \tag{A5}$$

In turn, the cardinality of the set α_n can be calculated using binary numbers for each scenario, as follows

$$\lambda_{sn} = \begin{cases} 1 & \text{If } \xi_s < \xi_n \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ s \in S, \quad 1 \le n \le N \tag{A6}$$

Then,

$$Card(\alpha_n) = \sum_s (\lambda_{s,n+1} - \lambda_{s,n}) \tag{A7}$$

Using this result together with Eq. A5 and the definition of p_s, Eq. A4 becomes

$$\begin{aligned}\int_{-\infty}^{\Omega} f(x, \xi)d\xi &= \lim_{NS\to\infty} \sum_{n=1}^{N\Omega} \sum_s p_s(\lambda_{s,n+1} - \lambda_{s,n}) \\ &= \lim_{NS\to\infty} \sum_s p_s \sum_{n=1}^{N\Omega} (\lambda_{s,n+1} - \lambda_{s,n})\end{aligned} \tag{A8}$$

Now the possible values for the last summation in the above equation are

$$\sum_{n=1}^{N\Omega} (\lambda_{s,n+1} - \lambda_{s,n}) = \begin{cases} 1 & \text{If } \xi_s < \Omega \\ 0 & \text{otherwise} \end{cases} \qquad \forall\ s \in S \tag{A9}$$

Notice that Eq. A9 coincides with the definition of z_s given in Eq. 10. Then, using this observation to replace the last summation into Eq. A8 the result sought is finally achieved

$$\int_{-\infty}^{\Omega} f(x, \xi)d\xi = \lim_{NS\to\infty} \sum_s p_s z_s \tag{A10}$$

B. Relation between expected profit and financial risk for scenario-based cases

This appendix derives the relationship between expected profit and financial risk for the case where the profit has a discrete probability distribution obtained from a finite number of scenarios. Assume that the two-stage stochastic problem with *NS* scenarios is solved obtaining the resulting profit for each scenario, ξ_s. Therefore, the expected value of the profit and the financial risk are given by

$$\mathrm{E}[Profit(x)] = \sum_{s\in S} p_s\xi_s \tag{B1}$$

$$Risk(x, \xi) = \sum_{s\in S} p_s z_s(x, \xi) \tag{B2}$$

Afterward, the scenarios were sorted in ascending profit order, such that $\xi_{s+1} \ge \xi_s$. Additionally, for the last scenario [the one with the highest profit and $Ordinal(s) = NS$] ξ_{NS+1} is defined as $\bar{\xi}$, with $\bar{\xi}$ being any value larger than ξ_{NS}. Similarly, $\bar{\xi}$ is defined as the lowest scenario profit [$Ordinal(s) = 1$]. Using these definitions, the incurred financial risk at a profit ξ_s can be expressed as

$$Risk^+(x, \xi_s) = Risk^+(x, \xi_{s-1}) + p_s \tag{B3}$$

Here, the superscript "+" stands for the limit coming from the right, given that in the scenario-based cases $Risk(x, \xi)$ is a discontinuous step-shaped function at ξ_s for all $s \in S$. Solving for p_s and replacing in Eq. B1 yields

$$\begin{aligned}\mathrm{E}[Profit(x)] = \sum_{\substack{s\in S\\ Ord(s)>1}} &[Risk^+(x, \xi_s) - Risk^+(x, \xi_{s-1})]\xi_s \\ &+ Risk^+(x, \underline{\xi})\underline{\xi}\end{aligned} \tag{B4}$$

By distributing the summations and rearranging, this last expression becomes

$$\begin{aligned}\mathrm{E}[Profit(x)] = &\left[\sum_{\substack{s\in S\\ Ord(s)>1}} Risk^+(x, \xi_s)\xi_s + Risk^+(x, \underline{\xi})\underline{\xi}\right] \\ &- \sum_{\substack{s\in S\\ Ord(s)>1}} Risk^+(x, \xi_{s-1})\xi_s\end{aligned} \tag{B5}$$

Notice that the term between parentheses in Eq. B5 is equal to $\sum_{s\in S} Risk^+(x, \xi_s)\xi_s$. In turn, the last term can be written as

$$\sum_{\substack{s\in S\\ Ord(s)<NS}} Risk^+(x, \xi_s)\xi_{s+1}$$

or simply

$$\sum_{s \in S} Risk^{+}(x, \xi_s)\xi_{s+1} - \bar{\xi}$$

Finally, using these expressions in Eq. B5, the relationship sought is obtained:

$$E[Profit(x)] = \bar{\xi} - \sum_{s \in S} Risk^{+}(x, \xi_s)(\xi_{s+1} - \xi_s) \tag{87}$$

The summation in this expression represents the area under the cumulative risk curve, as illustrated in Figure 8.

C. Behavior of the risk curves

Theorem 1 *Let x^* denote the optimal values of the first-stage variables for problem SP and x the values of first-stage variables for any other feasible solution with $E[Profit(x)] < E[Profit(x^*)]$ and $E[Profit(x^*)] < \infty$. Then, there exists $\Omega \in \Re$ such that $Risk(x, \Omega) > Risk(x^*, \Omega)$.*

Proof: The proof is by contradiction. Consider the case where the profit has a continuous probability distribution, given that the discrete distribution case can be seen as a particular case of the former. Assume that there exists a solution x for which $Risk(x, \Omega) \le Risk(x^*, \Omega)\ \forall\ \Omega \in \Re$. Then, the strategy is to show that this yields $E[Profit(x)] \ge E[Profit(x^*)]$, which is impossible because x^* is the optimal solution to model SP. Under these assumptions

$$Risk(x, \Omega) = \int_{-\infty}^{\Omega} f(x, \xi)d\xi \tag{C1}$$

$$Risk(x^*, \Omega) = \int_{-\infty}^{\Omega} f(x^*, \xi)d\xi \tag{C2}$$

$$E[Profit(x)] = \lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} \xi f(x, \xi)d\xi = \lim_{\alpha \to \infty} \int_{Risk(x,-\alpha)}^{Risk(x,\alpha)} \xi dRisk(x, \xi) \tag{C3}$$

$$E[Profit(x^*)] = \lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} \xi f(x^*, \xi)d\xi = \lim_{\alpha \to \infty} \int_{Risk(x^*,-\alpha)}^{Risk(x^*,\alpha)} \xi dRisk(x^*, \xi) \tag{C4}$$

Integrating by parts, the expected values can be rewritten as

$$E[Profit(x)] = \lim_{\alpha \to \infty} \left[\xi Risk(x, \xi)\Big|_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} Risk(x, \xi)d\xi \right] \tag{C5}$$

$$E[Profit(x^*)] = \lim_{\alpha \to \infty} \left[\xi Risk(x^*, \xi)\Big|_{-\alpha}^{\alpha} - \int_{-\alpha}^{\alpha} Risk(x^*, \xi)d\xi \right] \tag{C6}$$

The difference between the expected values is

$$E[Profit(x^*)] - E[Profit(x)] = \lim_{\alpha \to \infty} \left\{ \xi[Risk(x^*, \xi) - Risk(x, \xi)]\Big|_{-\alpha}^{\alpha} + \int_{-\alpha}^{\alpha} [Risk(x, \xi) - Risk(x^*, \xi)]d\xi \right\} \tag{C7}$$

The first term in the righthand side of Eq. C7 vanishes, given that $Risk(x^*, \alpha) = Risk(x, \alpha) = 1$ and $Risk(x^*, -\alpha) = Risk(x, -\alpha) = 0$ for some sufficiently large value of α. In addition, because of the assumption that $Risk(x, \Omega) \le Risk(x^*, \Omega)\ \forall\ \Omega \in \Re$, the integrand in the second term is negative $\forall\ \xi \in \Re$. Thus, one arrives at $E[Profit(x)] \ge E[Profit(x^*)]$, which is the contradiction sought.

Corollary 1 *It is impossible to obtain a feasible design x having a risk curve that lies entirely below the risk curve of the optimal design of problem SP.*

D. Equivalence of models RO-SP-FR and RR-SO-FR

Theorem 2 *For a given $\varepsilon_i \ge 0$, (x^*, y^*, z^*) is an optimal solution of problem RR-SP-FR if and only if there exist $\rho_i \ge 0$ such that (x^*, y^*, z^*) is an optimal solution of problem RO-SP-FR.*

Proof: The approach for this proof was first presented in Ahmed and Sahinidis (1998), applied to the robustness formulations. Consider the following Lagrangian relaxation (LR) of problem RR-SP-FR where constraints 35 are relaxed and ρ_i are the corresponding Lagrange multipliers.

LR

$$\text{Max} \sum_{s \in S} p_s q_s^T y_s - c^T x - \sum_{i \in I} \rho_i \left(\sum_{s \in S} p_s z_{si} - \varepsilon_i \right) \tag{88}$$

s.t.

Constraints 2 to 5
Constraints 30 to 32

Because the set of constraints 35 is convex, the strong duality theorem proves that a solution (x^*, y^*, z^*) is optimal to RR-SP-FR if and only if it is optimal to the above relaxed problem (LR) with $\rho_i \ge 0$. The proof follows by realizing that RO-SP-FR is equivalent to LR for positive ρ_i.

E. Pareto optimality of the solutions of risk management models

Theorem 3 *An optimal solution to models RO-SP-FR and RR-SP-FR is not stochastically dominated by any other solution; that is, the solution is Pareto optimal.*

Proof: Assume solution **I** to be an optimal solution to model RR-SP-FR. Furthermore, assume **I** is stochastically dominated by another feasible solution **II**. Then, by definition of stochastic dominance, **II** has an expected profit strictly greater than that of **I**. However, this contradicts the assumption that solution **I** is optimal. Thus, an optimal solution to RR-SP-FR cannot be stochastically dominated by any other solution. The proof for model RO-SP-FR follows from the equivalence derived in Theorem 2.

F. Feasible region of problems SP and RO-SP-FR

Theorem 4 *Any first-stage solution x that is feasible for problem SP with Profit(x)* $< \infty$, *is also feasible in problem RO-SP-FR; that is, the problems have the same feasible region.*

Proof: Clearly, if solution x is feasible in problem SP it must satisfy constraints 2 to 5, which are also present in problem RO-SP-FR. In addition, constraints 30 to 32 are satisfied for any value of x provided U_s is a valid profit upper bound. Consequently, all constraints of problem RO-SP-FR are satisfied and, thus, x is feasible in RO-SP-FR. Then, if every feasible solution to problem SP is also feasible in problem RO-SP-FR, it follows that the feasible region of the latter includes the feasible region of the former. In turn, any feasible solution of problem RO-SP-FR has to satisfy constraints 2 to 5 and is therefore feasible in SP. From this it follows that both feasible regions are equal.

Manuscript received Aug. 19, 2003, and revision received Aug. 21, 2003.